

USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020

Zoom Lecture: F: 2:00-4:00 p.m.

National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319

Dr. Steven L. Richardson (srichards22@comcast.net)

Professor Emeritus of Electrical Engineering, Department of Electrical and Computer Engineering, Howard University, Washington, DC

and

Faculty Associate in Applied Physics, John A. Paulson School of Engineering and Applied Sciences, Harvard University, Cambridge, MA

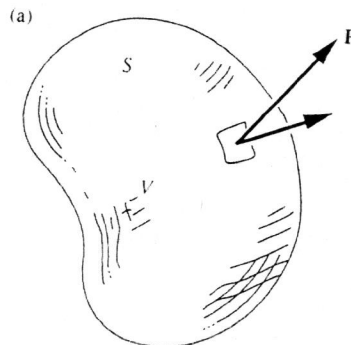
PROBLEM SET XII

(due Tuesday, September 15, 2020)

Problem 1

The electric field has a definite direction and magnitude at every point. It is a vector function of the coordinates which we have often indicated by writing $\vec{E}(x, y, z)$. What we are about to say can apply to any vector function, not just to the electric field and we shall use another symbol $\vec{F}(x, y, z)$ as a reminder of that. In other words, we shall talk mathematics rather than physics for a while and call \vec{F} simply a general vector function that we shall keep to three dimensions, however.

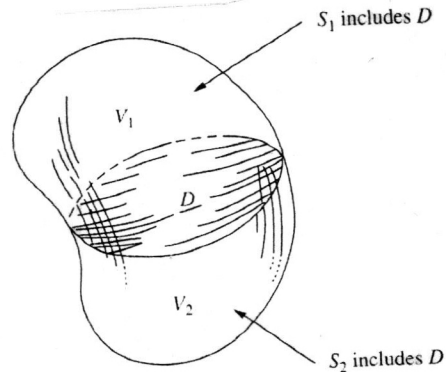
Consider a finite volume V of some shape, the surface of which we shall denote by a closed surface S . We are already familiar with the notion of the total flux Φ emerging from S . It is the value of the surface integral of \vec{F} extended over the whole of S



$$(1) \quad \Phi = \oint_S \vec{F} \cdot d\vec{S}$$

In the integrand $d\vec{S}$ is the infinitesimal vector whose magnitude is the area of a small element of S and whose direction is the outward-pointing normal to that little patch of surface as indicated in the previous figure.

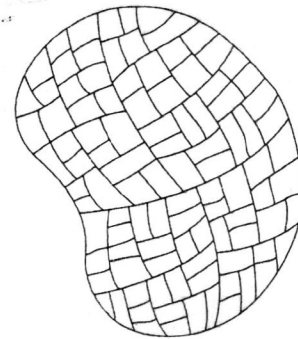
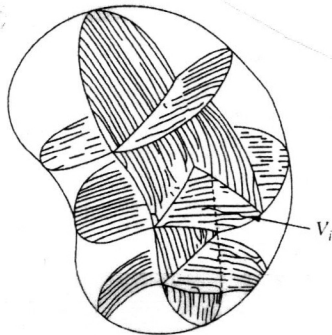
Now imagine dividing V into two parts by a surface or diaphragm D that cuts through the "balloon" S as illustrated below



Denote the two parts of V by V_1 and V_2 and, treating them as distinct volumes, compute the surface integral over each separately. The boundary surface S_1 of V_1 includes D , as so does S_2 . It is pretty obvious that the sum of the two surface integrals

$$(2) \quad \oint_{S_1} \vec{F} \cdot d\vec{S}_1 + \oint_{S_2} \vec{F} \cdot d\vec{S}_2$$

will equal the original integral over the whole surface expressed in Eq. (1). The reason is that any given patch on D contributes with one sign to the first integral and the same amount with opposite sign to the second, the "outward" direction in one case being the "inward" direction in the other. In other words, any flux out of V_1 , through the surface D , is flux into V_2 . The rest of the surface involved is identical to that of the original volume.



We can keep on subdividing until our internal partitions have divided V into a large number of parts, $V_1, \dots, V_i, \dots, V_N$, with surfaces $S_1, \dots, S_i, \dots, S_N$, as illustrated in the previous figures. No matter how far this is carried we can still be sure that

$$(3) \quad \sum_{i=1}^N \oint_{S_i} \vec{F} \cdot d\vec{S}_i = \oint_S \vec{F} \cdot d\vec{S} = \Phi$$

What we are after is this: in the limit as N becomes enormous we wish to identify something that is characteristic of a particular small region - and, ultimately, of the neighborhood of a point. Now the surface integral

$$(4) \quad \oint_{S_i} \vec{F} \cdot d\vec{S}_i$$

over one of the small regions is not such a quantity, for if we divide everything again, so that N becomes $2N$, this integral divides into two terms, each smaller than before since their sum is constant. In other words, as we consider smaller and smaller volumes in the same locality, the surface integral over one such volume gets steadily smaller. But we notice that, when we divide, the volume is also divided into two parts that sum to the original volume. This suggests that we look at the ratio of surface integral to volume for an element in the subdivided space

$$(5) \quad \frac{\oint_{S_i} \vec{F} \cdot d\vec{S}_i}{V_i}$$

It seems plausible that for N large enough, that is, for sufficiently fine-grained subdivision, we can halve the volume every time we halve the surface integral, so that we find, with continuing subdivision of any particular region, this ratio approaches a limit. If so, this limit is a property characteristic of the vector function \vec{F} in that neighborhood. We call it the divergence of \vec{F} , written $\text{div } \vec{F}$. That is, the value of $\text{div } \vec{F}$ at any point is defined as

$$(6) \quad \text{div } \vec{F} \equiv \lim_{V_i \rightarrow 0} \frac{1}{V_i} \oint_{S_i} \vec{F} \cdot d\vec{S}_i$$

where V_i is a volume including the point in question, and S_i , over which the surface integral is taken, is the surface of V_i . We must include the proviso that this limit exists and is independent of our method of subdivision. For the present we shall assume this is always true.

The meaning of $\text{div } \vec{F}$ can be expressed in this way: $\text{div } \vec{F}$ is the flux out of V_i per unit of volume, in the limit of infinitesimal V_i . It is a scalar quantity, obviously. It may vary from place to place, its value at any particular location (x, y, z) being the limit of the ratio in Eq. 6 as V_i is chopped smaller and smaller while always enclosing the point (x, y, z) . So $\text{div } \vec{F}$ is simply a scalar function of the coordinates. If we know this scalar function of position $\text{div } \vec{F}$ we can work our way right back to the surface integral over a large volume. We first write Eq. 3 in this way

$$(7) \quad \oint_S \vec{F} \cdot d\vec{S} = \sum_{i=1}^N \oint_{S_i} \vec{F} \cdot d\vec{S}_i = \sum_{i=1}^N V_i \left[\frac{\oint_{S_i} \vec{F} \cdot d\vec{S}_i}{V_i} \right]$$

In the limit $N \rightarrow \infty$, $V_i \rightarrow 0$, the term in brackets becomes the divergence of \vec{F} , and the sum goes into a volume integral

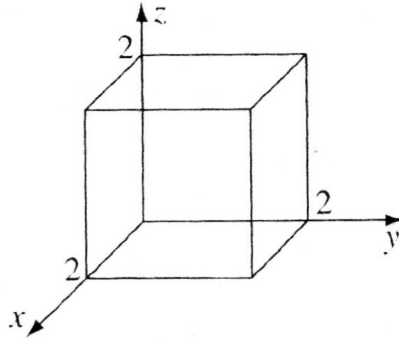
$$(8) \quad \oint_S \vec{F} \cdot d\vec{S} = \int_V \text{div } \vec{F} \, dV$$

This result is known as the Divergence Theorem and it holds for any vector field for which the limit involved in Eq. 6 exists. Note that the entire content of the theorem is contained in Eq. 3 which itself is simply the statement that the fluxes cancel in pairs over the interior boundaries of all the little regions. The other steps in the proof were the multiplication by 1 in the form of $\frac{V_i}{V_i}$, the use of the definition in Eq. 6, and the conversion of an infinite sum into an integral.

Now at last here is an application of the beautiful proof! Verify the Divergence Theorem for the function

$$(9) \quad \vec{v} = (xy) \hat{i} + (2yz) \hat{j} + (3zx) \hat{k}$$

and use as your volume the cube shown below with sides of length 2.



Problem 2

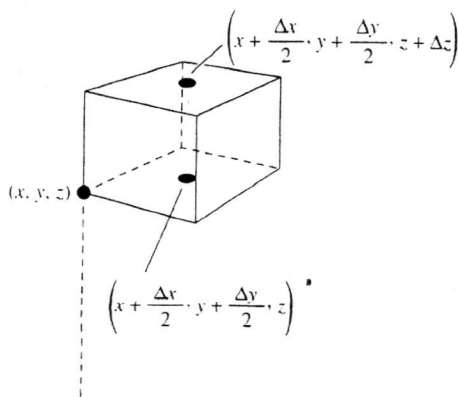
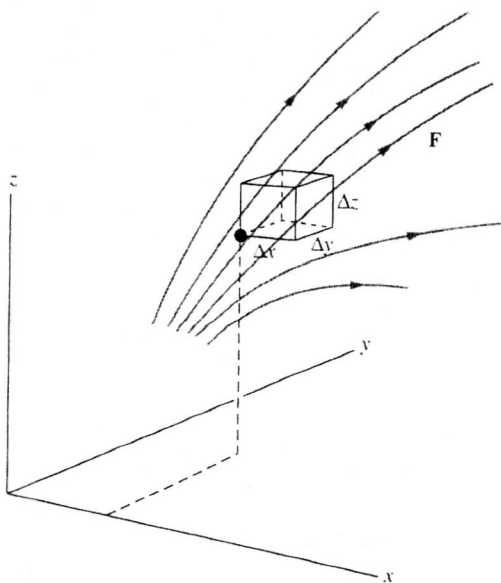
In Problem 1 we obtained the fundamental definition of $\text{div } \vec{F}$ without reference to any particular coordinate system as

$$(1) \quad \text{div } \vec{F} \equiv \lim_{V_i \rightarrow 0} \frac{1}{V_i} \oint_{S_i} \vec{F} \cdot d\vec{S}_i$$

We previously defined in this course the definition of $\text{div } \vec{F}$ in Cartesian coordinates as

$$(2) \quad \text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

To see how these two equations are connected let us explicitly calculate the divergence of a vector function F in Cartesian coordinates x , y , and z . That means we have three scalar functions $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$. Let us consider a rectangular box of volume with one corner at the point (x, y, z) and sides Δx , Δy , and Δz as shown in the figure below



Consider two opposite faces of the box, the top and bottom for instance, which would be represented by $d\vec{S}$ vectors $\hat{k} \Delta x \Delta y$ and $-\hat{k} \Delta x \Delta y$. The flux through these faces involves only the z -component of \vec{F} and the net contribution depends on the difference between F_z at the top and F_z at the bottom, or more precisely, on the difference between the average of F_z over the top face and the average of F_z over the bottom face of the box. To first order in small quantities this difference is

$$(3) \quad \Delta z \frac{\partial F_z}{\partial z}$$

and the previous figure will help illustrate this. The average of F_z on the bottom surface of the box if we consider only first-order variations in F_z over this small rectangle, is its value at the center of the rectangle. We can find this by looking at the beginning of a Taylor expansion of the scalar function $F_z(x, y, z)$ in the neighborhood of (x, y, z) . That is

$$F_z(x + a, y + b, z + c) = F_z(x, y, z) + \left(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} \right) F_z + \dots + \frac{1}{n!} \left(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} \right)^n F_z + \dots$$

The derivatives are all to be evaluated at (x, y, z) . In our case

$$(4) \quad a = \frac{\Delta x}{2}, \quad b = \frac{\Delta y}{2}, \quad c = 0$$

and to first order in the small displacements

$$(5) \quad F_z(x, y, z) + \frac{\Delta x}{2} \frac{\partial F_z}{\partial x} + \frac{\Delta y}{2} \frac{\partial F_z}{\partial y} +$$

For the average of F_z over the top face we take the value at the center of the top face, which up to first order in the small displacements is

$$(6) \quad F_z(x, y, z) + \frac{\Delta x}{2} \frac{\partial F_z}{\partial x} + \frac{\Delta y}{2} \frac{\partial F_z}{\partial y} + \Delta z \frac{\partial F_z}{\partial z}$$

The net flux out of the box through these two faces, each of which has the area $\Delta x \Delta y$, is therefore the difference between the flux out of the box at the top and the flux into the box at bottom

$$\Delta x \Delta y \left[F_z(x, y, z) + \frac{\Delta x}{2} \frac{\partial F_z}{\partial x} + \frac{\Delta y}{2} \frac{\partial F_z}{\partial y} + \Delta z \frac{\partial F_z}{\partial z} \right] - \Delta x \Delta y \left[F_z(x, y, z) + \frac{\Delta x}{2} \frac{\partial F_z}{\partial x} + \frac{\Delta y}{2} \frac{\partial F_z}{\partial y} \right]$$

which reduces to

$$(7) \quad \Delta x \Delta y \Delta z \frac{\partial F_z}{\partial z}$$

Obviously, similar statements must apply to the other pairs of sides. That is, the net flux out of the box is

$$(8) \quad \Delta x \Delta z \Delta y \frac{\partial F_y}{\partial y}$$

through the sides parallel to the xz -plane and

$$(9) \quad \Delta y \Delta z \Delta x \frac{\partial F_x}{\partial x}$$

through the sides parallel to the yz -plane. Note that the product

$$(10) \quad \Delta x \Delta y \Delta z$$

occurs in all of these expressions. Thus the total flux out of the little box is

$$(11) \quad \Phi = \Delta x \Delta y \Delta z \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right)$$

The volume of the box is

$$(12) \quad \Delta x \Delta y \Delta z$$

so the ratio of flux to volume is

$$(13) \quad \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

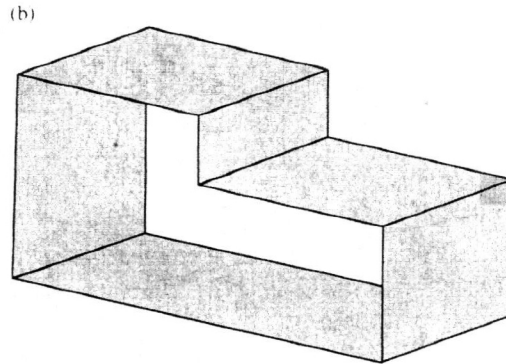
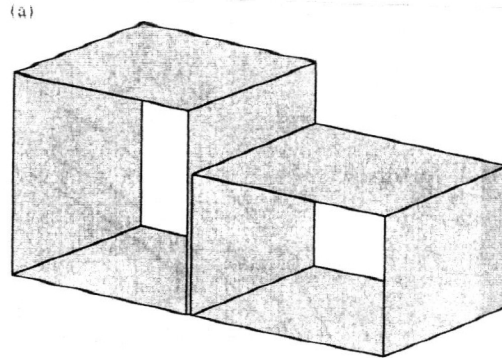
and as this expression does not contain the dimensions of the box at all it remains as the limit when we let the box shrink. Note that had we retained any terms proportional to

$$(14) \quad (\Delta x)^2, (\Delta x \Delta y) \dots$$

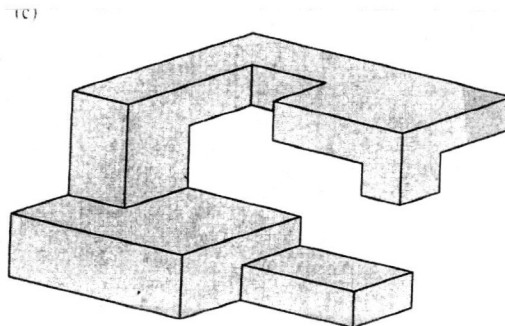
they would of course vanish on going to the limit.

Now we can begin to see why this limit is going to be independent of the shape of the box. Obviously it is independent of the proportions of the rectangular box, but that is not saying much. It is easy to see that it will be the same for any volume

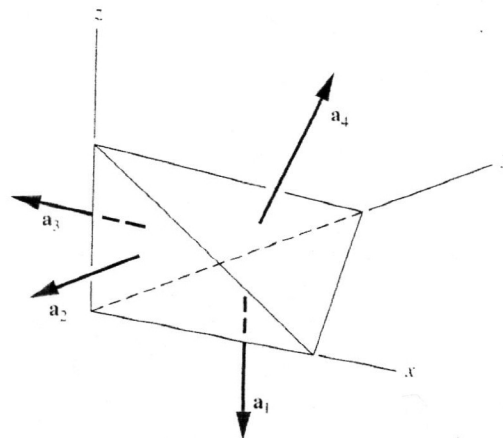
that we can make by sticking together little rectangular boxes of any size and shape. Consider the two boxes in the figure below



The sum of the flux Φ_1 out of box 1 and Φ_2 out of box 2 is not changed by removing the adjoining walls to make one box, for whatever flux went through that plane was negative flux for one and positive for the other. So we could have a bizarre shape like in the figure below



without affecting the result. We leave it to the reader to generalize further. Tilted surfaces can be taken care of if you first prove that the vector sum of the four surface areas of the tetrahedron shown below is zero



We conclude that, assuming only that the functions F_x , F_y , and F_z are differentiable, the limit does exist and it is given by

$$(15) \quad \operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

which is our expected result from our previous discussions in lecture for the components of the vector $\operatorname{div} \vec{F}$ in Cartesian coordinates.

Now verify the Divergence Theorem for the function (actually we saw a bit of this example in Problem Set III)

$$\vec{D} = \rho^2 \cos^2 \phi \hat{\rho} + z \sin \phi \hat{\phi}$$

where the volume V is a cylinder described by the following equations

$$\rho = 4$$

$$0 \leq z \leq 1$$

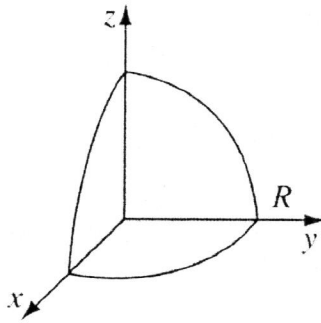
Problem 3

Given the vector function

$$(1) \quad \vec{G} = r^2 \cos \theta \hat{r} + r^2 \cos \phi \hat{\theta} - r^2 \cos \theta \sin \phi \hat{\phi}$$

prove the Divergence Theorem

$$(2) \quad \oint_S \vec{G} \cdot d\vec{S} = \int_V \text{div } \vec{G} \, dV$$

using your volume one octant of the sphere of radius R in the figure below.

Make sure you include the entire surface and note that your answer should be

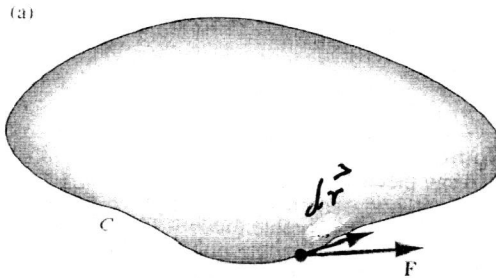
$$(3) \quad \frac{\pi R^4}{4}$$

Problem 4

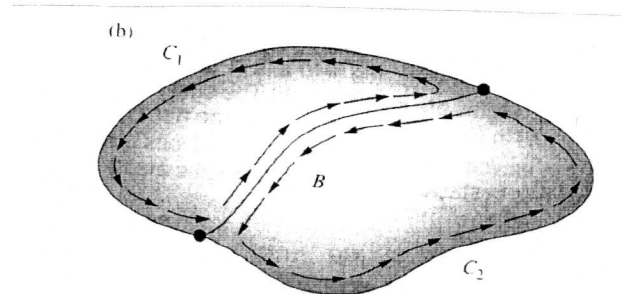
We developed the concept of the divergence, a local property of a vector field, by starting from the surface integral over a large closed surface. In the same spirit, let us consider the line integral of some vector field $\vec{F}(x, y, z)$, taken around a closed path some curve C that comes back to join itself. The curve C can be visualized as the boundary of some surface S that spans it. A good name for the magnitude of such a closed path line integral is circulation and we shall use Γ as the symbol

$$(1) \quad \Gamma = \oint_C \vec{F} \cdot d\vec{r}$$

In the integrand $d\vec{r}$ is the element of path, an infinitesimal vector locally tangent to C as shown below



There are two senses in which C could be traversed and we have to select one to make the direction of $d\vec{r}$ unambiguous. Incidentally, the curve C need not lie in a plane as it could be as crooked as you like. Now bridge C with a new path B , thus making two loops, C_1 and C_2 , each of which includes B as a part of itself



Take the line integral around each of these in the same directional sense. It is easy to see that the sum of these two circulations, Γ_1 and Γ_2 , will be the same as the original circulation about C . The reason is that the bridge is traversed in opposite directions to the two integrations, leaving just the contributions that made up the original line integral around C . Further subdivisions into many loops, $C_1, \dots, C_i, \dots, C_N$, leaves the sum unchanged

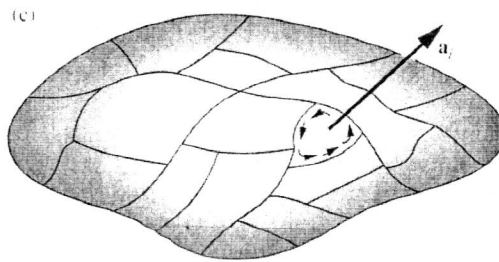
$$(2) \quad \oint_C \vec{F} \cdot d\vec{r} = \sum_{i=1}^N \oint_{C_i} \vec{F} \cdot d\vec{r}_i$$

or

$$(3) \quad \Gamma = \sum_{i=1}^N \Gamma_i$$

In the same manner as in our discussion of divergence in Problem 1, we can continue indefinitely to subdivide, now by adding new bridges instead of new surfaces, seeking in the limit to arrive at a quantity characteristic of the field \vec{F} in a local environment. When we subdivide the loops, we make loops with smaller circulation but with smaller area. So it is natural to consider the ratio of loop circulation to loop area, just as we considered in Problem 1 the ratio of flux to volume. However, things are a little different here because the area $d\vec{S}_i$ of the bit of surface that spans a small loop C_i is really a vector as shown below in contrast with the scalar volume V_i in Problem

1. A surface has an orientation in space, whereas a volume does not. In fact, as we make smaller and smaller loops in some neighborhood, we can arrange to have a loop oriented in any direction we choose as shown in the figure below



Remember, we are not committed to any particular surface over the whole curve C . Thus we can pass to the limit in essentially many different ways and we must expect the result to reflect this.

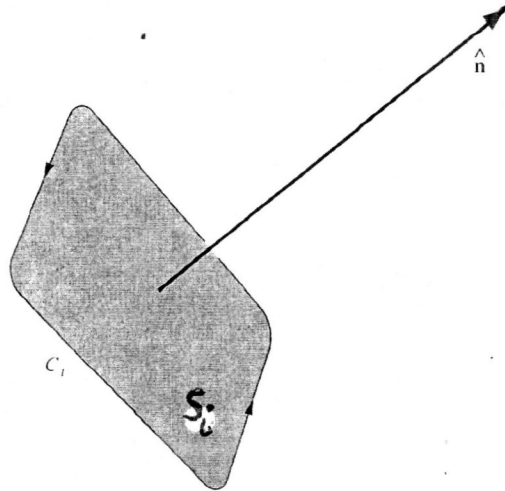
Let us choose some particular orientation for the patch as it goes through the last stages of subdivision. The unit vector \hat{n} will denote the normal to the patch, which is to remain fixed in direction as the patch surrounding a particular point \vec{P} shrinks down toward zero size. Remember points are vectors and vectors are points! The limit of the ratio of circulation to patch area will be written as

$$(4) \quad \lim_{S_i \rightarrow 0} \frac{\Gamma_i}{S_i}$$

or

$$(5) \quad \lim_{S_i \rightarrow 0} \frac{\oint_{C_i} \vec{F} \cdot d\vec{r}}{S_i}$$

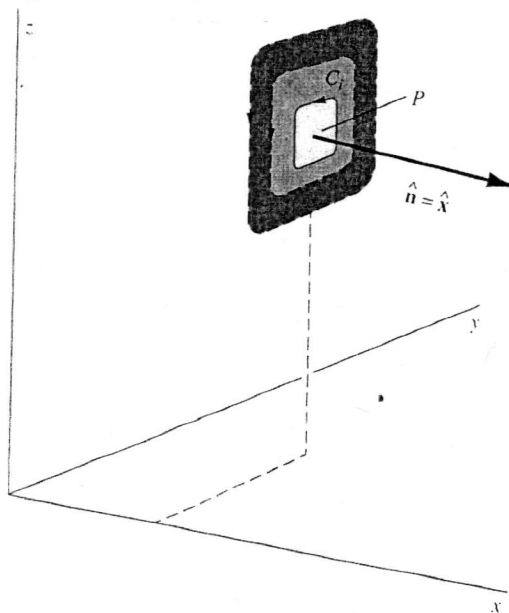
The rule for the sign is that the direction of \hat{n} and the sense in which C_i is traversed in the line integral shall be related by the right-hand rule as shown below



The limit we obtain by this procedure is a scalar quantity that is associated with the point \vec{P} in the vector field \vec{F} and with the direction of \hat{n} . We could pick three directions such as \hat{i} , \hat{j} , and \hat{k} , and get three different numbers. It turns out that these numbers can be considered components of a vector. We call that vector $\text{curl } \vec{F}$. That is to say, the number we obtain for the limit with \hat{n} in a particular direction is the component, in that direction, of the vector $\text{curl } \vec{F}$. To state this in an equation

$$(6) \quad (\text{curl } \vec{F}) \cdot \hat{n} = \lim_{S_i \rightarrow 0} \frac{\oint_{C_i} \vec{F} \cdot d\vec{r}}{S_i}$$

where \hat{n} is the unit normal vector to the curve C_i . For instance, the x-component of $\text{curl } \vec{F}$ is obtained by choosing $\hat{n} = \hat{i}$ as shown in the figure below



As the loop shrinks down around the point \vec{P} , we keep it in a plane perpendicular to the x-axis. In general, the vector $\text{curl } \vec{F}$ will vary from place to place. If we let the patch shrink down around some other point the ratio of circulation to area may have a different value, depending on the nature of the vector function $\text{curl } \vec{F}$. That is, $\text{curl } \vec{F}$ is itself a vector function of the coordinates. Its direction at each point in space is normal to the plane through this point in which the circulation is a maximum. Its magnitude is the limiting value of circulation per unit area, in this plane, around the point in question.

The last two sentences might be taken as a definition of $\text{curl } \vec{F}$. Like Eq. 6 they make no reference to a coordinate frame. We have not proved that the object so named and defined is in fact a vector; we have only asserted it. Possession of direction and magnitude is not enough to make something a vector. The components as defined must behave like vector components. Suppose we have determined certain values for the x, y, and z components of $\text{curl } \vec{F}$ by applying Eq. 6 with \hat{n} chosen, successively as \hat{i} , \hat{j} , and \hat{k} . If $\text{curl } \vec{F}$ is a vector then it is uniquely determined by these three components. If some fourth direction is now chosen for \hat{n} , the left side of Eq. 6 is fixed and the quantity on the right, the circulation in the plane perpendicular to the new \hat{n} , had better agree with it! Indeed, until one is sure that $\text{curl } \vec{F}$ is a vector, it is not even obvious that there can be at most one direction for which the circulation per unit area at \vec{P} is a maximum - as was tacitly assumed in the latter definition. In fact, Eq. 6 does define a vector, but we shall not give a proof of that here.

Now we finally are in a position to put all of this together and obtain something important, like Stokes' Theorem! From the circulation around an infinitesimal patch of surface we can now work back to the circulation around the original large loop C

$$(7) \quad \Gamma = \oint_C \vec{F} \cdot d\vec{S} = \sum_{i=1}^N \Gamma_i = \sum_{i=1}^N S_i \left(\frac{\Gamma_i}{S_i} \right)$$

In the last step we merely multiplied and divided by S_i . Now observe what happens to the right-hand side as N is made enormous and all the S_i area shrink. From Eq. 6, the quantity in parentheses becomes $(\text{curl } \vec{F}) \cdot \hat{n}_i$, where \hat{n}_i is the unit vector normal to the i^{th} patch. So we have on the right the sum, over all patches that make up the entire surface S spanning C , of the product "patch area times the normal component of $\text{curl } \vec{F}$." This is simply the surface integral over S of the vector $\text{curl } \vec{F}$

$$(8) \quad \sum_{i=1}^N S_i \left(\frac{\Gamma_i}{S_i} \right) = \sum_{i=1}^N S_i (\text{curl } \vec{F}) \cdot \hat{n}_i \rightarrow \oint_S \text{curl } \vec{F} \cdot d\vec{S}$$

because $d\vec{S} = S_i \hat{n}_i$ by definition. We find that

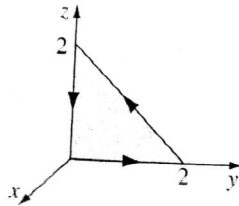
$$(9) \quad \oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot d\vec{S}$$

This is Stokes' Theorem. Note how it resembles the Divergence Theorem, in structure. Stokes' Theorem relates the line integral of a vector to the surface integral of the curl of the vector. The Divergence Theorem in Problem 1 relates the surface integral of a vector to the volume integral of the divergence of the vector. Stokes' Theorem involves a surface and the closed curve that bounds it. The Divergence Theorem involves a volume and the surface that encloses it.

Now at last here is an application of this beautiful proof! Verify Stokes' Theorem for the function

$$\vec{v} = (xy)\hat{i} + (2yz)\hat{j} + (3zx)\hat{k}$$

using the triangular shaded area in the figure below



Problem 5

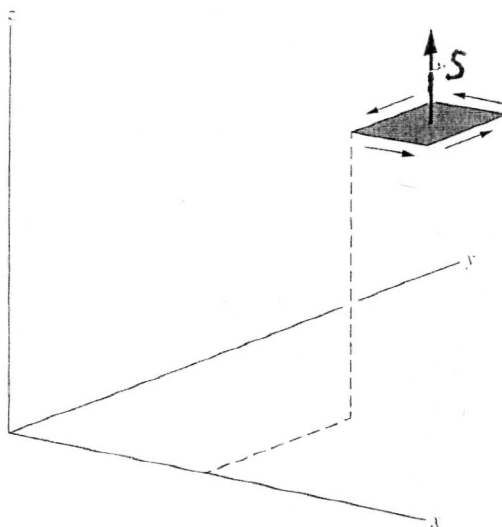
In Problem 4 we obtained the fundamental definition of $\text{curl } \vec{F}$ without reference to any particular coordinate system as

$$(1) \quad (\text{curl } \vec{F}) \cdot \hat{n} = \lim_{S_i \rightarrow 0} \frac{\oint_{C_i} \vec{F} \cdot d\vec{r}}{S_i}$$

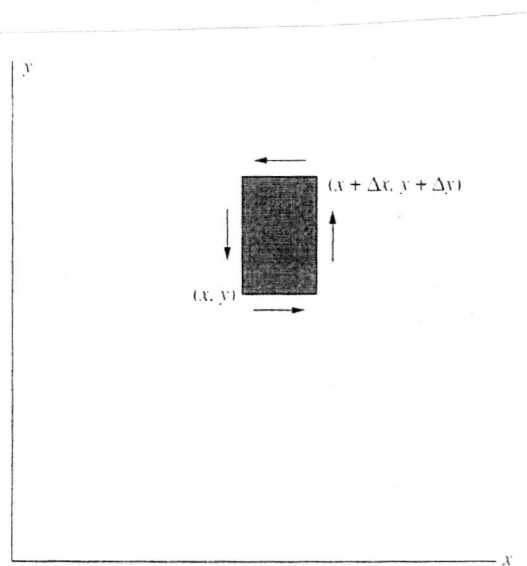
We previously defined in this course the definition of $\text{curl } \vec{F}$ in Cartesian coordinates as

$$(2) \quad \text{curl } \vec{F} = \hat{i} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{j} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{k} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

To see how these two equations are connected let us carry out the integration called for in Eq. 6 of Problem 4, but we will do it over a path of a very simple shape, one that encloses a rectangular patch of surface parallel to the xy -plane as illustrated below



That is we are taking the $\hat{n} = \hat{k}$. In agreement with our rule about sign, the direction of integration around the rim must be clockwise as seen by someone looking up in the direction of \hat{n} . In the figure below we look down onto the rectangle from above



The line integral of \vec{F} around such a path depends on the variation of F_x with y and the variation of F_y with x . For if F_x had the same average value along the top of the frame, as shown in the previous figure, as along the bottom of the frame, the contribution of these two pieces of the whole line integral would obviously cancel. A similar remark applies to the side members. To first order in the small quantities Δx and Δy , the difference between the average F_x over the top segment of path at $y + \Delta y$ and its average over the bottom segment at y is

$$(3) \quad \left(\frac{\partial F_x}{\partial y} \right) \Delta y$$

This follows from an argument similar to the one we used in Problem 2 where at the midpoint of the bottom of the frame

$$(4) \quad F_x = F_x(x, y) + \frac{\Delta x}{2} \frac{\partial F_x}{\partial x}$$

while at the midpoint of the top of the frame

$$(5) \quad F_x = F_x(x, y) + \frac{\Delta x}{2} \frac{\partial F_x}{\partial x} + \frac{\Delta y}{\partial y} \frac{\partial F_x}{\partial y}$$

These are just the average values referred to in first order in the Taylor expansion. It is their difference times the length of the path segment Δx that determines their net contribution to the circulation. This circulation is

$$(6) \quad -\Delta x \Delta y \frac{\partial F_x}{\partial y}$$

The minus sign comes in because we are integrating toward the left at the top, so that if F_y is more positive at the top it results in a negative contribution to the circulation. The contribution from the sides is

$$(7) \quad \Delta y \Delta x \frac{\partial F_y}{\partial x}$$

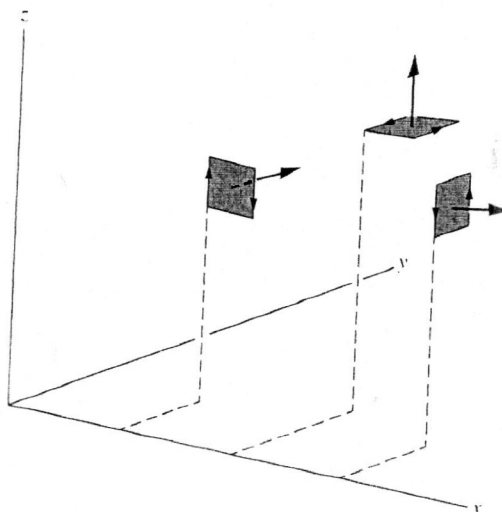
and here the sign is positive, because if F_y is more positive on the right the result is a positive contribution to the circulation. Thus neglecting any higher powers of Δx and Δy the line integral around the whole rectangle is

$$(8) \quad \oint \vec{F} \cdot d\vec{r} = -\Delta x \cdot \left(\frac{\partial F_x}{\partial y} \right) \Delta y + \Delta y \cdot \left(\frac{\partial F_y}{\partial x} \right) \Delta x = \Delta x \Delta y \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

Now $\Delta x \Delta y$ is the magnitude of the area of the enclosed rectangle, which we have represented by a vector in the z -direction. Evidently the quantity

$$(9) \quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

is the limit of the ratio of the line integral around the patch to the area of the patch as the patch shrinks to zero size. If the rectangular frame had been oriented with its normal in the positive y-direction, like the left frame in the figure below



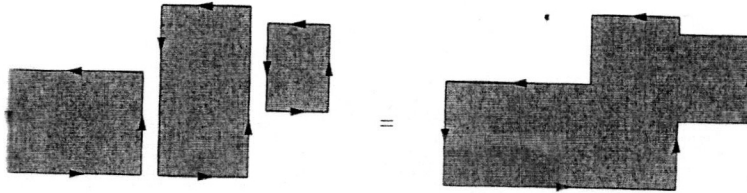
we would have found the expression

$$(10) \quad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}$$

for the limit of the corresponding ratio. Similarly, if the frame had been oriented with its normal in the positive x-direction, like the right frame in the figure above, we would have obtained

$$(11) \quad \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

Although we have considered rectangles only, our result is actually independent of the shape of the little patch and its frame, for reasons much the same as in the case of the integrals involved in the Divergence Theorem. For instance, it is clear that we can freely join different rectangles to form other figures because the line integrals along the merging sections of the boundary cancel one another exactly as shown below



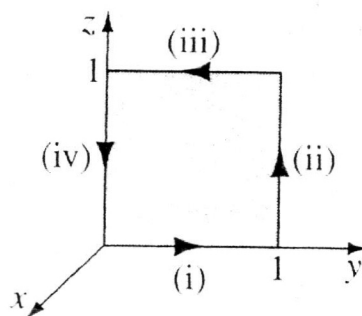
We conclude that, for any of these orientations, the limit of the ratio of circulation to area is independent of the shape of the patch we choose. Thus we obtain the result we expected from our previous discussions in lecture for the components of the vector $\text{curl } \vec{F}$ in Cartesian coordinates

$$(12) \quad \text{curl } \vec{F} = \hat{i} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{j} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{k} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

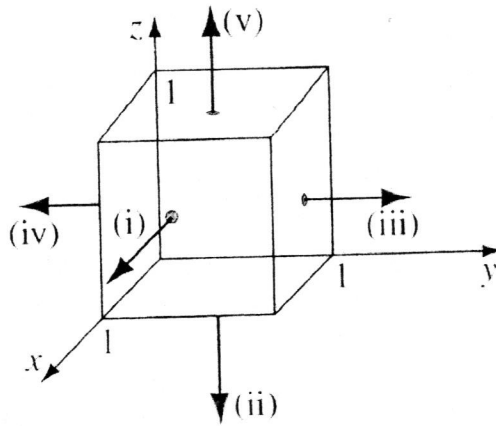
Now at last here is an application of application of Stokes' Theorem. In Lecture 12 we showed that for the function

$$(13) \quad \vec{v} = (2xz + 3y^2) \hat{j} + (4yz^2) \hat{k}$$

using the square surface shown in the figure below that Stokes' Theorem was true.



It is important to realize that the surface integral in Stokes' Theorem depends only on the boundary line C of the surface and not the particular surface S . Use this observation to redo this problem by integrating over the five faces of the cube in the figure below using the same vector function and the same boundary line as done in Lecture 12. Note that the back of the cube is open.



Problem 6

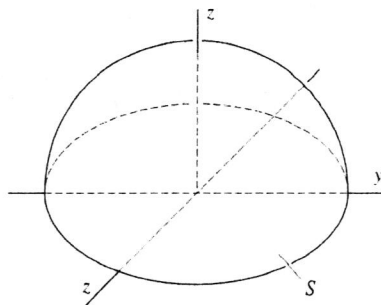
Given the vector function

$$(1) \quad \vec{F} = z \hat{i} + x \hat{j} - x \hat{k}$$

prove Stokes' Theorem

$$(2) \quad \oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot d\vec{S}$$

where C is a circle of radius 1 centered at the origin and lying in the xy -plane and S is the hemisphere shown below



Problem 7

Given the Divergence Theorem listed below for any vector field F

$$(1) \quad \oint_S \vec{F} \cdot d\vec{S} = \int_V \text{div } \vec{F} \, dV$$

let us see what it implies for the electric field E . Recall that Gauss's Law can be written as

$$(2) \quad \oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_V \rho \, dV$$

If the Divergence Theorem holds for any vector field, it certainly holds for \vec{E}

$$(3) \quad \oint_S \vec{E} \cdot d\vec{S} = \int_V \text{div } \vec{E} \, dV$$

If we compare Eqs. (2) and (3) we realize that they must hold for any volume we care to choose of any shape, size, or location. Comparing them, we see that this can only be true if, at every point

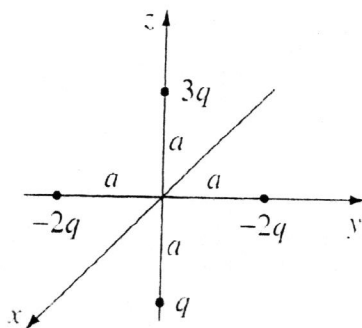
$$(4) \quad \text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

which is Gauss's Law in differential form, that is, stated in terms of a local relation between charge density and electric field. It is also known as Maxwell's First Equation.

Let us now return to a problem we have solved before in the course: the electric field \vec{E} of a sphere of radius R and uniform volume charge density ρ . Use Maxwell's First Equation to verify the expected results for \vec{E} both inside and outside the sphere.

Problem 8

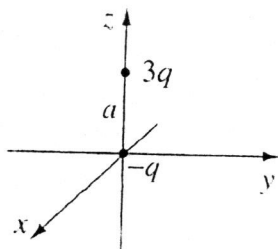
Four particles (one of charge q , one of charge $3q$, and two of charge $-2q$) are placed as shown in the figure below each a distance a from the origin.



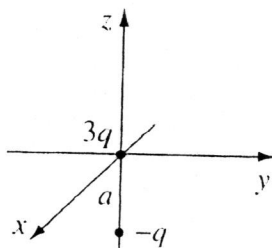
Find a simple approximate formula for the potential, valid at points far from the origin. Express your answer in spherical polar coordinates.

Problem 9

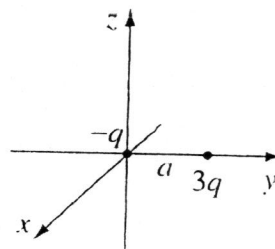
Two point charges, $3q$ and $-q$, are separated by a distance a . For each of the arrangements in the figure below, find (i) the monopole moment, (ii) the dipole moment, and (iii) the approximate potential in spherical polar coordinates at large r , including both the monopole and dipole contributions.



(a)



(b)



(c)

Problem 10: All Good Things Must Come to an End

We have reached the end of this short on-line summer course, "Using Vector Calculus to Solve Problems in Electricity and Magnetism". I want to acknowledge the many textbooks I have freely and liberally drawn from to develop both my lectures and problem sets. I list these references by starting with the textbook I have most frequently consulted and going in descending order from there. I will not make any editorial comments on the pros and cons of each textbook, but if you are interested in my personal viewpoints, please directly contact me by email!

1. David J. Griffiths, "Introduction to Electrodynamics", 4th ed., Pearson, Boston, MA, 2013.
2. Matthew N. O. Sadiku, "Elements of Electromagnetics", 7th ed., Oxford University Press, New York, 2018.
3. Edward M. Purcell and David J. Morin, "Electricity and Magnetism", 3rd ed., Cambridge University Press, 2013.
4. Harry M. Schey, "Div, Grad, Curl, and All That: An Informal Text on Vector Calculus", 4th ed., W. W. Norton, New York, 2005.
5. Donald A. McQuarrie, "Mathematical Methods for Scientists and Engineers", University Science Books, 2003.
6. Paul Lorrain and Dale R. Corson "Electromagnetic Fields and Waves", 2nd ed., W. H. Freeman, 1970.
7. Andrew Zangwill, "Modern Electrodynamics", Cambridge University Press, 2013.
8. John D. Jackson, "Classical Electrodynamics", 3rd ed. John Wiley & Sons, New York, 2007.

Problem 1

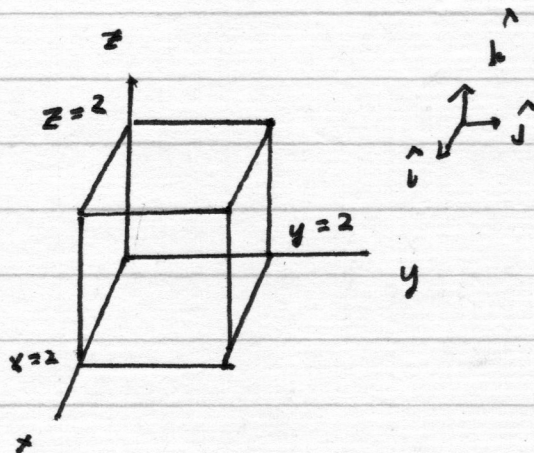
Given the Divergence Theorem

$$\int_V \text{div } \vec{F} dV = \oint_S \vec{F} \cdot d\vec{S}$$

test it for the vector function \vec{F}

$$\vec{F} = xy\hat{i} + 2yz\hat{j} + 3zx\hat{k}$$

where your volume V is the cube below with sides of length 2.



We see that

S_1 (front) $\hat{n} = \hat{i}$ $dS_1 = dy dz$

S_2 (back) $\hat{n} = -\hat{i}$ $dS_2 = dy dz$

S_3 (left) $\hat{n} = -\hat{j}$ $dS_3 = dx dz$

$$S_4 \text{ (right)} \quad \hat{n} = +\hat{j} \quad dS_4 = dx dz$$

$$S_5 \text{ (top)} \quad \hat{n} = \hat{k} \quad dS_5 = dx dy$$

$$S_6 \text{ (bottom)} \quad \hat{n} = -\hat{k} \quad dS_6 = dx dy$$

$$\oint_S \vec{F} \cdot d\vec{S} = \sum_{i=1}^6 \oint_{S_i} \vec{F} \cdot d\vec{S}_i$$

We tackle each of our six surface integrals one at a time

$$\int_{S_1} \vec{F} \cdot d\vec{S} = \int_{S_1} xy \, dS_1 = \iint xy \, dy \, dz$$

For S_1 $x = 2$

$$\int_0^2 dz \int_0^2 2y \, dy = 2 \left. \frac{2y^2}{2} \right|_0^2 = 8$$

$$\int_{S_2} \vec{F} \cdot d\vec{S} = - \int xy \, dS_1 = - \int xy \, dy \, dz$$

For S_2 $x = 0$

$$\int_{S_2} \vec{F} \cdot d\vec{S} = 0$$

$$\int_{S_3} \vec{F} \cdot d\vec{S} = - \iint 2yz \, dx \, dz$$

For S_3 $y = 0$

$$\int_{S_3} \vec{F} \cdot d\vec{S} = 0$$

$$\int_{S_4} \vec{F} \cdot d\vec{S} = + \iint 2yz \, dx \, dz$$

For S_4 $y = 2$

$$\int 4z \, dx \, dz = 4 \int_0^2 dx \int_0^2 z \, dz$$

$$\int_{S_4} \vec{F} \cdot d\vec{S} = 4 \cdot 2 \cdot \left. \frac{z^2}{2} \right|_0^2 = 16$$

$$\int_{S_5} \vec{F} \cdot d\vec{S} = \int_{S_5} 3zx \, dx \, dy$$

For S_5 $z = 2$

$$= \iint 6x \, dx \, dy = 6 \int_0^2 x \, dx \int_0^2 dy = 12 \cdot \left. \frac{x^2}{2} \right|_0^2$$

$$= 24$$

$$\int_{S_5} \vec{F} \cdot d\vec{S} = 24$$

$$\int_{S_6} \vec{F} \cdot d\vec{S} = - \int 3zx \, dx \, dy$$

For S_6 $z = 0$

$$\int_{S_6} \vec{F} \cdot d\vec{S} = 0$$

$$\oint_S \vec{F} \cdot d\vec{S} = 8 + 0 + 0 + 16 + 24 + 0 = 48$$

Now let us find

$$\int_V \text{div} \vec{F} \, dV$$

For $\vec{F} = \hat{i}xy + \hat{j}2yz + 3zx\hat{k}$

$$\text{div} \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = y + 2z + 3x$$

$$\int_V \text{div} \vec{F} \, dV = \int_V y \, dV + 2 \int_V z \, dV + 3 \int_V x \, dV$$

In Cartesian Coordinates $dV = dx \, dy \, dz$

Now let us tackle each of these integrals one at a time

$$\begin{aligned}\int_V y \, dx \, dy \, dz &= \int y \, dy \int dx \int dz \\ &= \frac{y^2}{2} \Big|_0^2 \times \Big|_0^2 \times z \Big|_0^2 \\ &= \frac{4}{2} \cdot 2 \cdot 2 = 8\end{aligned}$$

$$\begin{aligned}2 \int_V z \, dV &= 2 \int z \, dz \int dx \int dy \\ &= 2 \frac{z^2}{2} \Big|_0^2 \times \Big|_0^2 \times y \Big|_0^2 = 4 \cdot 2 \cdot 2 = 16\end{aligned}$$

$$\begin{aligned}3 \int_V x \, dV &= 3 \int x \, dx \int dy \int dz \\ &= 3 \frac{x^2}{2} \Big|_0^2 \times \Big|_0^2 \times z \Big|_0^2 = \frac{6 \cdot 2 \cdot 2 \cdot 2}{2} =\end{aligned}$$

Thus

24

$$\int_V \text{Div } \vec{F} \, dV = 8 + 16 + 24 = \oint_S \vec{F} \cdot d\vec{S} = 48$$

and our Divergence Theorem works in this example for \vec{F} !

Problem 2

Verify that

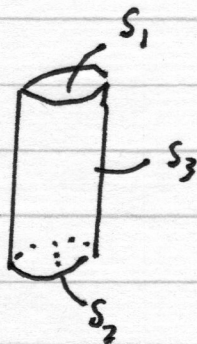
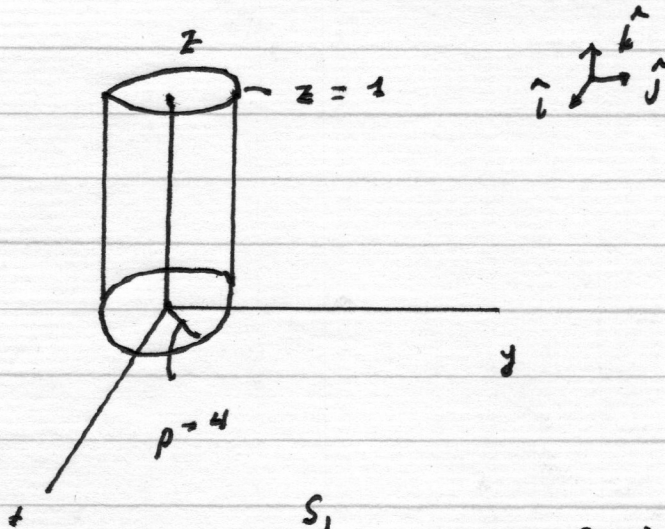
$$\underline{\vec{D} = \rho^2 \cos^2 \phi \hat{\rho} + z \sin \phi \hat{\phi}}$$

over the closed surface of the cylinder

$$\underline{0 \leq z \leq 1}$$

$$\underline{\rho = 4}$$

satisfies the Divergence Theorem.



$$\oint_S \vec{D} \cdot d\vec{s} = \int_{S_1} \vec{D} \cdot d\vec{s}_1 + \int_{S_2} \vec{D} \cdot d\vec{s}_2 + \int_{S_3} \vec{D} \cdot d\vec{s}_3$$

Let us handle S_3 first

For S_3

$$d\vec{S} = \hat{\rho} \rho d\phi dz \quad [\text{from Problem Set III}]$$

so

$$\vec{D} \cdot d\vec{S} = \rho^3 d\phi dz \cos^2 \phi$$

and

$$\int_{S_3} \vec{D} \cdot d\vec{S} = \rho^3 \int_0^1 dz \int_0^{2\pi} \cos^2 \phi d\phi$$

We can use the following half-angle identity to solve the last integral

$$2 \cos^2 \phi - 1 = \cos 2\phi$$

$$\cos^2 \phi = \frac{1 + \cos 2\phi}{2}$$

$$\int_0^{2\pi} \cos^2 \phi d\phi = \int_0^{2\pi} \frac{d\phi}{2} + \frac{1}{2} \int_0^{2\pi} \cos 2\phi d\phi$$

$$= \frac{2\pi}{2} + \frac{1}{4} \int \cos u du,$$

$$\int_0^{2\pi} \cos^2 \phi d\phi = \pi + \frac{1}{4} \sin(2\phi) \Big|_0^{2\pi} = \pi$$

$u = 2\phi; du = 2d\phi$

Thus

$$\int_{S_3} \vec{D} \cdot d\vec{S} = \rho^3 \pi = 64\pi$$

Now to tackle the remaining integrals

$$\int_{S_1} \vec{D} \cdot d\vec{S}_1$$

$$d\vec{S}_1 = k \hat{\rho} \rho d\rho d\phi$$

$$\vec{D} \cdot d\vec{S}_1 = 0$$

$$\int_{S_2} \vec{D} \cdot d\vec{S}_2$$

$$d\vec{S}_2 = -k \hat{\rho} \rho d\rho d\phi$$

$$\vec{D} \cdot d\vec{S}_2 = 0$$

Thus

$$\oint_S \vec{D} \cdot d\vec{S} = 64\pi$$

Now let us deal with the volume integral

$$\int_V \text{div } \vec{D} dV$$

$$\text{div } \vec{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

(in cylindrical polar coordinates)

$$D_\rho = \rho^2 \cos^2 \phi$$

$$D_\phi = z \sin \phi$$

$$D_z = 0$$

$$\text{div } \vec{D} = \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} [\rho^3 \cos^2 \phi] \right) + \frac{1}{\rho} z \frac{\partial \sin \phi}{\partial \phi}$$

+ 0

$$\text{div } \vec{D} = \frac{1}{\rho} 3\rho^2 \cos^2 \phi + \frac{z}{\rho} \cos \phi$$

$$\text{div } \vec{D} = 3\rho \cos^2 \phi + \frac{z}{\rho} \cos \phi$$

$$\int_V \text{div } \vec{D} dV = \int_V 3\rho \cos^2 \phi dV + \int_V \frac{z}{\rho} \cos \phi dV$$

1st Integral

$$dV = \rho dp d\phi dz$$

$$\int_V 3\rho \cos^2\phi \rho dp d\phi dz$$

$$= \int_0^4 3\rho^2 d\rho \int_0^1 dz \int \cos^2\phi d\phi$$

$$= \frac{3\rho^3}{3} \Big|_0^4 z \Big|_0^1 (\pi), \text{ where we have}$$

previously seen this
last integral

$$= 4^3 \pi = 64\pi$$

2nd Integral

$$\int_V \frac{z}{\rho} \cos\phi \rho dp d\phi dz$$

$$= \int z dz \int d\rho \int \cos\phi d\phi$$

$$= \frac{z^2}{2} \Big|_0^1 \rho \Big|_0^4 \sin\phi \Big|_0^{2\pi} = 0$$

Thus

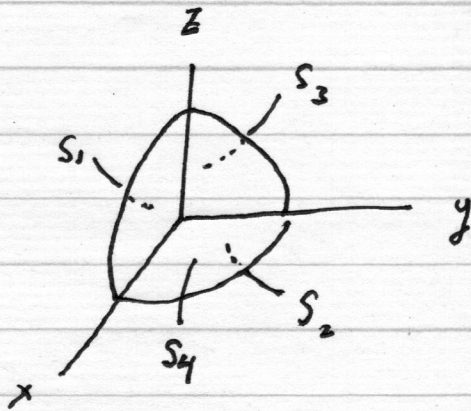
$$\oint_S \vec{D} \cdot d\vec{S} = \int_V \text{div } \vec{D} dV = 64\pi \text{ and all is well!}$$

Problem 3

Check the Divergence Theorem for the function

$$\vec{c} = r^2 \cos \theta \hat{r} + r^2 \cos \phi \hat{\theta} - r^2 \cos \theta \sin \phi \hat{\phi}$$

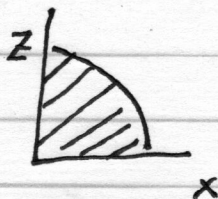
Using as your volume one octant of the sphere of radius R. Make sure you include the entire surface.



$$\oint_S \vec{c} \cdot d\vec{S} = \sum_{j=1}^4 \int_{S_j} \vec{c} \cdot d\vec{S}_j$$

For S_1

$$dS = r dr d\theta$$



$$d\vec{S} = -\hat{j} dS$$

$$\phi = 0$$

$$\int_{S_1} \vec{C} \cdot d\vec{S} = - \int_{S_1} \vec{C} \cdot \hat{j} dS$$

In spherical polar coordinates (Problem Set I)

$$\hat{j} = \hat{r} \sin \theta \sin \phi + \hat{\theta} \cos \theta \sin \phi + \hat{\phi} \cos \phi$$

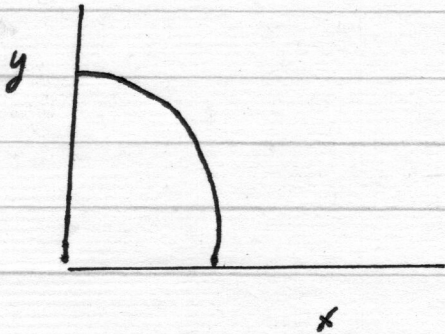
$$\hat{j} \cdot \hat{j} = \sin^2 \theta \sin^2 \phi + \cos^2 \theta \sin^2 \phi + \cos^2 \phi = 1$$

$$\begin{aligned} \hat{j} \cdot \vec{C} &= \sin \theta \sin \phi r^2 \cos \theta + \cos \theta \sin \phi r^2 \cos \phi \\ &\quad - \cos \phi r^2 \cos \theta \sin \phi \end{aligned}$$

Since $\phi = 0$ $\sin \phi = 0$ and all terms vanish

$$\int_{S_1} \vec{C} \cdot d\vec{S} = 0$$

For S_2



$$dS = r dr d\phi$$

$$\theta = \pi/2$$

$$\vec{dS} = -\hat{k} dS = -\hat{k} r dr d\phi$$

$$\int_{S_2} \vec{C} \cdot \vec{dS} = - \int_{S_2} \vec{C} \cdot \hat{k} r dr d\phi$$

$$\left. \begin{aligned} \hat{k} &= \hat{r} \cos \theta - \hat{\theta} \sin \theta \\ \hat{k} \cdot \hat{k} &= \cos^2 \theta + \sin^2 \theta = 1 \end{aligned} \right\} \begin{array}{l} \text{Problem} \\ \text{Set I} \end{array}$$

$$\vec{C} \cdot \hat{k} = (\hat{r} \cos \theta - \hat{\theta} \sin \theta) \cdot$$

$$(\hat{r} \cos \theta + \hat{\phi} \sin \theta - \hat{\theta} \sin \theta)$$

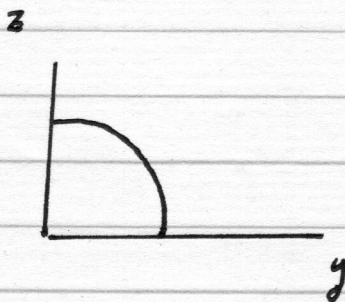
$$\vec{C} \cdot \hat{k} = r^2 \cos^2 \theta - r^2 \cos \phi \sin \theta - r^2 \cos \theta \sin \phi \hat{\phi}$$

and since $\theta = \pi/2$

$$\vec{C} \cdot \hat{k} = -r^2 \cos \phi$$

$$\int_{S_2} \vec{c} \cdot d\vec{S} = + \int_{S_2} r^2 \cos \phi \, r \, dr \, d\phi$$

$$\begin{aligned} \int_{S_2} \vec{c} \cdot d\vec{S} &= \int_0^R r^3 \, dr \int_0^{\pi/2} \cos \phi \, d\phi \\ &= \frac{R^4}{4} \end{aligned}$$

For S_3 

$$dS = r \, dr \, d\theta$$

$$\phi = \pi/2$$

$$d\vec{S} = -\hat{i} \, r \, dr \, d\theta$$

$$\int_{S_3} \vec{c} \cdot d\vec{S} = - \int_{S_3} \vec{c} \cdot \hat{i} \, r \, dr \, d\theta$$

From Problem Set I

$$\hat{i} = \hat{r} \sin \theta \cos \phi + \hat{\theta} \cos \theta \cos \phi - \hat{\phi} \sin \theta$$

$$\hat{i} \cdot \hat{i} = \sin^2 \theta \cos^2 \phi + \cos^2 \theta \cos^2 \phi + \sin^2 \theta$$

-15-

$$\vec{C} \cdot \hat{i} \quad (r^2 \cos \theta \hat{r} + r^2 \cos \phi \hat{\theta} - r^2 \cos \theta \sin \phi \hat{\phi})$$

$$\cdot (\hat{r} \sin \theta \cos \phi + \hat{\theta} \cos \theta \cos \phi - \hat{\phi} \sin \phi)$$

$$\vec{C} \cdot \hat{i} = r^2 \cos \theta \sin \theta \cos \phi$$

$$+ r^2 \cos \phi \cos \theta \cos \phi + r^2 \cos \theta \sin^2 \phi$$

Since $\phi = \pi/2 \Rightarrow \cos \pi/2 = 0$; $\sin \phi =$
 $\sin \pi/2 = 1$

$$\vec{C} \cdot \hat{i} = r^2 \cos \theta$$

$$\int_{S_3} \vec{C} \cdot d\vec{S} = - \int_{S_3} \vec{C} \cdot \hat{i} r dr d\theta$$

$$= - \int r^3 \cos \theta dr d\theta$$

$$\int_{S_3} \vec{C} \cdot d\vec{S} = - \int_0^R r^3 dr \int_0^{\pi/2} \cos \theta d\theta$$

$$\int_{S_3} \vec{b} \cdot d\vec{S} = -\frac{R^4}{4}$$

Finally, for S_4

$$dS = r^2 \sin \theta d\theta d\phi \quad \left[\text{Check Problem set III} \right]$$

$$d\vec{S} = \hat{r} r^2 \sin \theta d\theta d\phi$$

$$\vec{b} \cdot d\vec{S} = r^2 \cos \theta r^2 \sin \theta d\theta d\phi$$

$$\int_{S_4} \vec{b} \cdot d\vec{S} = \int_{S_4} R^4 \sin \theta \cos \theta d\theta d\phi$$

(N.B. r is a constant R)

$$\int_{S_4} \vec{b} \cdot d\vec{S} = R^4 \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi$$

$$\int_{S_4} \vec{b} \cdot d\vec{S} = \frac{R^4 \pi}{2} \int_0^{\pi/2} \sin \theta \cos \theta d\theta$$

To do our integral

$$\int_0^{\pi/2} \sin \theta \cos \theta d\theta$$

$$u = \sin \theta$$

$$du = \cos \theta d\theta$$

$$\int_0^1 u du = \frac{u^2}{2} \Big|_0^1 = \frac{1}{2}$$

θ	u
0	0
$\frac{\pi}{2}$	1

$$\int_{S_4} \vec{b} \cdot d\vec{S} = \frac{\pi R^4}{2} \frac{1}{2} = \frac{\pi R^4}{4}$$

Thus

$$\oint_S \vec{b} \cdot d\vec{S} = 0 + \frac{R^4}{4} - \frac{R^4}{4} + \frac{\pi R^4}{4}$$

$$\oint_S \vec{b} \cdot d\vec{S} = \frac{\pi R^4}{4}$$

Now let us compute the volume integral

$$\int_V \operatorname{div} \vec{c} \, dV$$

In spherical polar coordinates

$$\begin{aligned} \vec{\nabla} \cdot \vec{c} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 c_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta c_\theta) \\ &\quad + \frac{1}{r \sin \theta} \left(\frac{\partial c_\phi}{\partial \phi} \right) \end{aligned}$$

$$\text{Term I} \quad \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r^2 \cos \theta)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} (r^4 \cos \theta) = \frac{4r^3}{r^2} \cos \theta = 4r \cos \theta$$

$$\text{Term II} \quad \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta r^2 \cos \phi)$$

$$= \frac{1}{r \sin \theta} \cos \theta r^2 \cos \phi$$

$$\text{Term III} \quad \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \phi} [-r^2 \cos \theta \sin \phi] \right)$$

$$= \frac{-\cos \phi \cos \theta r^2}{r \sin \theta}$$

$$\vec{\nabla} \cdot \vec{\sigma} = 4r \cos \theta + \frac{\cos \theta r^2 \cos \phi}{r \sin \theta} - \frac{\cos \theta r^2 \cos \phi}{r \sin \theta}$$

$$\vec{\nabla} \cdot \vec{\sigma} = 4r \cos \theta$$

$$\int_V \vec{\nabla} \cdot \vec{\sigma} dV = \int_V 4r \cos \theta r^2 dr \sin \theta d\theta d\phi$$

$$\int_V \vec{\nabla} \cdot \vec{\sigma} dV = 4 \int_0^R r^3 dr \int_0^{\pi/2} \sin \theta d\theta \cos \theta \int_0^{\pi/2} d\phi$$

$$\int_V \text{div } \vec{\sigma} dV = \frac{4R^4}{4} \frac{\pi}{2} \int_0^{\pi/2} \sin \theta \cos \theta d\theta$$

We have previously
done this integral

$$\Rightarrow \frac{1}{2}$$

$$\int_V \text{div } \vec{\sigma} dV = \frac{\pi R^4}{4}$$

$$= \oint_S \vec{\sigma} \cdot d\vec{S} = \frac{\pi R^4}{4}$$

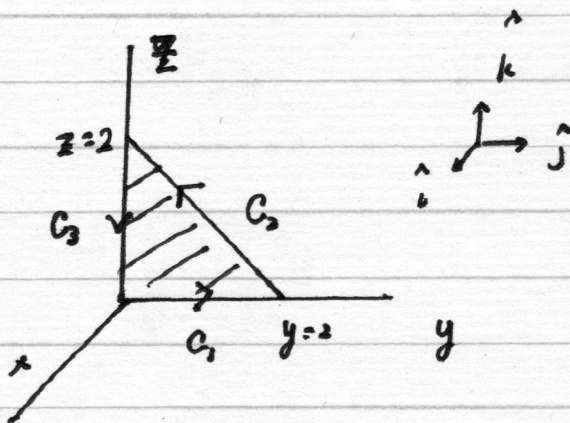
and the Divergence Theorem is proven!

Problem 4

Verify Stokes' Theorem for the vector function

$$\underline{\vec{v} = (xy)\hat{i} + (2yz)\hat{j} + (3zx)\hat{k}}$$

Using the triangular shaded area below



Stokes' Theorem

$$\oint_C \vec{v} \cdot d\vec{r} = \int_S (\text{curl } \vec{v}) \cdot d\vec{S}$$

$$\oint_C \vec{v} \cdot d\vec{r} = \int_{C_1} \vec{v} \cdot d\vec{r} + \int_{C_2} \vec{v} \cdot d\vec{r} + \int_{C_3} \vec{v} \cdot d\vec{r}$$

Let

$$d\vec{r} = \hat{i} dx + \hat{j} dy + \hat{k} dz$$

$$\vec{v} \cdot d\vec{r} = (xy) dx + (2yz) dy + (3zx) dz$$

$$\int_{C_1} \vec{v} \cdot d\vec{r} \quad \text{Along } C_1 \quad \begin{array}{l} x=0, dx=0 \\ z=0; dz=0 \end{array}$$

~~$\int_{C_1} (2yz) dy$~~

$$\int_{C_1} \vec{v} \cdot d\vec{r} = 0$$

$$\int_{C_2} \vec{v} \cdot d\vec{r} \quad \text{Along } C_2 \quad \begin{array}{l} x=0, dx=0 \\ z = -y + 2 \end{array}$$

$$\int_C (2yz) dy = \int_2^0 2y(-y+2) dy$$

$$= \int_2^0 -2y^2 dy + \int_2^0 4y dy$$

$$= \left. -\frac{2}{3}y^3 \right|_2^0 + \left. \frac{4y^2}{2} \right|_2^0$$

$$\int_{C_2} \vec{v} \cdot d\vec{r} = \frac{2}{3} \cdot 8 - 2 \cdot 4 = \frac{16}{3} - 8 = \frac{16}{3} - \frac{24}{3} = -\frac{8}{3}$$

$$\int_{C_3} \vec{v} \cdot d\vec{r} \quad \text{Along } C_3 \quad y=0, x=0, dy=0$$

$$\int_{C_3} \vec{v} \cdot d\vec{r} = 0$$

Then

$$\oint_C \vec{v} \cdot d\vec{r} = -8/3$$

Now let us deal with the surface integr.

$$\text{curl } \vec{v} = \hat{i} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{j} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{k} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$$

$$\frac{\partial v_z}{\partial y} = 0$$

$$\frac{\partial v_y}{\partial z} = 2y$$

$$\frac{\partial v_x}{\partial z} = 0$$

$$\frac{\partial v_z}{\partial x} = 3z$$

$$\frac{\partial v_y}{\partial x} = 0$$

$$\frac{\partial v_x}{\partial y} = x$$

$$\text{curl } \vec{v} = \hat{i}(-2y) + \hat{j}(-3z) - \hat{k}x$$

$$\int_S (\text{curl } \vec{v}) \cdot d\vec{S}$$

From our figure

$$d\vec{S} = \hat{i} dS = \hat{i} dy dz$$

$$\int_S (\text{curl } \vec{v}) \cdot d\vec{S} = \int_S -2y dy dz$$

$$= -2 \int \left[\int y dy \right] dz$$

Recall that y and z are not independent variables since $y = 2 - z$ so the inner integral has to be done carefully

$$\int_0^{2-z} y dy = \left. \frac{y^2}{2} \right|_0^{2-z} = \frac{(2-z)^2}{2}$$

Now we can evaluate the outer integral

$$\int_S (\text{curl } \vec{v}) \cdot d\vec{S} = -2 \int_0^2 \frac{(2-z)^2}{2} dz$$

$$\int_S (\text{curl } \vec{v}) \cdot d\vec{S} = -\frac{2}{2} \int_0^2 (4 - 4z + z^2) dz$$

$$= \left[-4z + \frac{4z^2}{2} - \frac{z^3}{3} \right]_0^2$$

$$= \left[-8 + 2 \cdot 4 - \frac{8}{3} \right] = -\frac{8}{3}$$

Thus we have shown that Stokes' Theorem works

$$\int_S (\text{curl } \vec{v}) \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{r} = -\frac{8}{3}$$

Problem 5

Check that

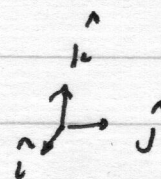
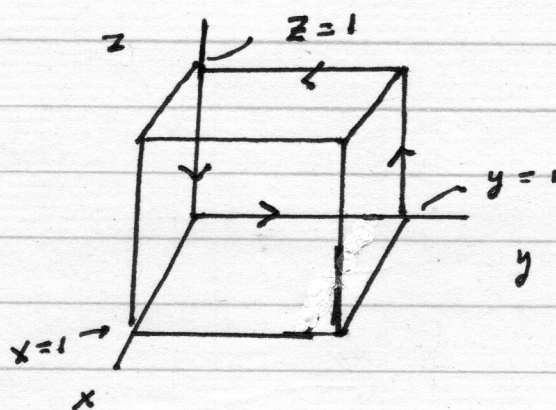
$$\int_S (\text{curl } \vec{v}) \cdot d\vec{S}$$

depends only on the boundary line and not on the particular surface by using

$$\vec{v} = (2xz + 3y^2)\hat{j} + (4yz^2)\hat{k}$$

and the boundary line below, but integrating over the five faces of the cube shown below.

Note that the back of the cube is open



- S_1 front
- S_2 bottom
- S_3 right
- S_4 left
- S_5 top

Note that the open surface S (back face) is bounded by C

$$\int_S (\text{curl } \vec{v}) \cdot d\vec{s} = \sum_{i=1}^5 \int_{S_i} (\text{curl } \vec{v}) \cdot d\vec{s}_i$$

$$\text{curl } \vec{v} = \hat{i} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{j} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{k} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$$

Since $v_x = 0$

$$\vec{\nabla} \times \vec{v} = \hat{i} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{j} \frac{\partial v_z}{\partial x}$$

$$+ \hat{k} \frac{\partial v_y}{\partial x}$$

$$\frac{\partial v_z}{\partial y} = 4z^2$$

$$\frac{\partial v_y}{\partial z} = 2x$$

$$\frac{\partial v_z}{\partial x} = 0$$

$$\frac{\partial v_y}{\partial x} = 2z$$

$$\boxed{\vec{\nabla} \times \vec{v} = \hat{i} (4z^2 - 2x) + \hat{k} 2z}$$

Now let us tackle each of the five surface integrals

$$\int_{S_1} (\text{curl } \vec{v}) \cdot d\vec{S}$$

For S_1

$$\hat{n} = +\hat{k}$$

$$dS = dy dz$$

$x=1$

$$= \int_{S_1} (4z^2 - 2x) dy dz$$

$$= \iint 4z^2 dz dy - 2 \iint dy dz$$

$$= \left. \frac{4z^3}{3} \right|_0^1 \Big|_0^1 - 2 \left. y \right|_0^1 \Big|_0^1$$

$$= \frac{4}{3} - 2 = \frac{4}{3} - \frac{6}{3} = -\frac{2}{3}$$

$$\int_{S_2} (\text{curl } \vec{v}) \cdot d\vec{S} \quad \text{For } S_2 \quad \hat{n} = -\hat{k}$$

$$z=0 \quad dz=0$$

$$\int_{S_2} (\text{curl } \vec{v}) \cdot d\vec{S} = - \int 2z dx dy = 0 \quad dS = dx dy$$

$$\int_{S_3} (\text{curl } \vec{v}) \cdot d\vec{S}$$

For S_3 $\hat{n} = +\hat{j}$

$$y = 1 \quad dy = 0$$

$$dS = dx dz$$

$$\int_{S_3} (2xz + 3y^2) dx dz$$

$$\int_{S_3} 2xz dx dz + \int_{S_3} 3 dx dz$$

$$2 \frac{x^2}{2} \Big|_0^1 \frac{z^2}{2} \Big|_0^1 + 3x \Big|_0^1 z \Big|_0^1$$

$$= \frac{1}{2} + 3 = \frac{1}{2} + \frac{6}{2} = \frac{7}{2}$$

$$\int_{S_4} (\text{curl } \vec{v}) \cdot d\vec{S}$$

For S_4 $\hat{n} = -\hat{j}$

$$y = 0 \quad dy = 0$$

$$dS = dx dz$$

$$\int_{S_4} (\text{curl } \vec{v}) \cdot d\vec{S} = - \int_{S_4} (2xz + 3y^2) dx dz = -\frac{7}{2}$$

$$\int_{S_5} (\text{curl } \vec{v}) \cdot d\vec{S}$$

For S_5 $\hat{n} = +\hat{k}$

$$dS = dx dy \quad ; \quad z = 1$$

$$dz = 0$$

$$\int_{S_5} (\text{curl } \vec{v}) \cdot d\vec{S} = \iint 4y \, dx \, dy = 4 \left. \frac{y^2}{2} \right|_0^1 \left. x \right|_0^1$$

$$= 2$$

Thus

$$\int_S (\text{curl } \vec{v}) \cdot d\vec{S} = -\frac{2}{3} + 0 + \frac{7}{2} - \frac{7}{2} + 2$$

$$= \frac{6}{3} - \frac{2}{3} = \frac{4}{3}$$

Finally, as we proved in Lecture 12

$$\oint_C \vec{v} \cdot d\vec{r} = \frac{4}{3}$$

so

$$\int_S (\text{curl } \vec{v}) \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{r} = \frac{4}{3}$$

and Stokes' Theorem works!

Problem 6

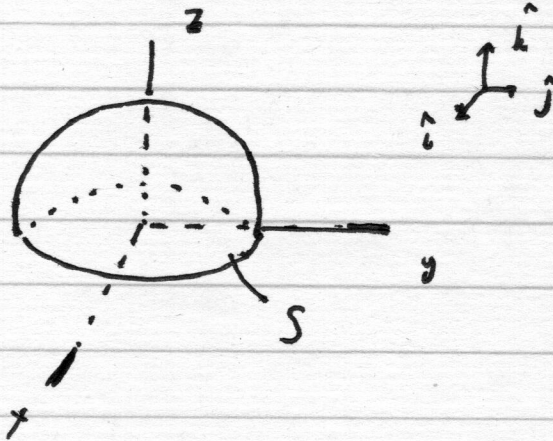
Given the vector function

$$\underline{\vec{F}} = \hat{i}z + \hat{j}x - \hat{k}x$$

prove Stokes' Theorem

$$\int_S (\text{curl } \vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

where C is a circle of radius 1 centered at the origin and lying in the xy-plane and S is the hemisphere shown below



$$\vec{F} = \hat{i} z + \hat{j} x - \hat{k} x$$

$$\vec{\nabla} \times \vec{F} = \hat{i} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{j} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{k} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

$$\vec{\nabla} \times \vec{F} = ?$$

$$\frac{\partial F_z}{\partial y} = 0$$

$$\frac{\partial F_y}{\partial z} = 0$$

$$\frac{\partial F_x}{\partial z} = 1$$

$$\frac{\partial F_z}{\partial x} = -1$$

$$\frac{\partial F_y}{\partial x} = 1$$

$$\frac{\partial F_x}{\partial y} = 0$$

$$\vec{\nabla} \times \vec{F} = 0 - 0 + \hat{j} (1 - (-1)) + \hat{k} (0 + 1)$$

$$\vec{\nabla} \times \vec{F} = \hat{k} + 2\hat{j}$$

$$\int_S (\text{curl } \vec{F}) \cdot d\vec{S} = (\hat{k} + 2\hat{j}) \cdot d\vec{S}$$

$$d\vec{S} = \hat{r} dS$$

$$dS = r^2 \sin \theta d\theta d\phi$$

$$\hat{r} = \hat{i} \sin \theta \cos \phi + \hat{j} \sin \theta \sin \phi$$

$$+ \hat{k} \cos \theta \quad \left[\text{From Problem} \right]$$

set I

1st Integral

$$\int \hat{k} \cdot d\vec{S} = \int r^2 \sin \theta d\theta d\phi \cos \theta$$

$$r=1 \quad \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi$$

$$u = \sin \theta$$

$$du = \cos \theta d\theta$$

$$= \frac{u^2}{2} \Big|_0^{\pi/2} \cdot 2\pi$$

$$= \frac{1}{2} \sin^2 \theta \Big|_0^{\pi/2} \cdot 2\pi = \pi$$

2nd Integral

$$2 \int \hat{j} \cdot d\vec{S} = \int \hat{j} \cdot \hat{r} dS$$
$$= \int r^2 \sin \theta d\theta d\phi \sin \theta \sin \phi$$

$$= \int_0^{2\pi} d\phi \sin \phi \int_0^{\pi/2} \sin^2 \theta d\theta$$
$$= \left[-\cos \phi \right]_0^{2\pi} \int_0^{\pi/2} \sin^2 \theta d\theta$$

$$= 0$$

Thus

$$\int_S (\text{curl } \vec{F}) \cdot d\vec{S} = \pi$$

Now let us deal with the line integral

$$\int_C \vec{F} \cdot d\vec{r} = \int (z\hat{u} + j\hat{x} - k\hat{x}) \cdot d\vec{r}$$

In plane polar coordinates

$$d\vec{r} = \rho d\phi \hat{\phi}$$

$$\hat{\phi} = -\hat{u} \sin \phi + \hat{j} \cos \phi$$

$$\int (z\hat{u} + j\hat{x} - k\hat{x}) \rho d\phi (-\hat{u} \sin \phi + \hat{j} \cos \phi)$$

$$\int \rho d\phi [-z \sin \phi + x \cos \phi]$$

In xy-plane $z = 0$ and $\rho = r = 1$
 $x = r \cos \phi = \cos \phi$

$$\int_0^{2\pi} d\phi \cos^2 \phi$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \cos^2 \phi$$

Using a half-angle identity

$$\cos^2 \phi = \frac{1 + \cos 2\theta}{2}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left[\frac{1}{2} + \frac{\cos 2\theta}{2} \right] d\theta$$

$$= \frac{2\pi}{2} + \frac{1}{2} \int_0^{2\pi} \cos 2\theta d\theta$$

$$u = 2\theta$$

$$du = 2 d\theta$$

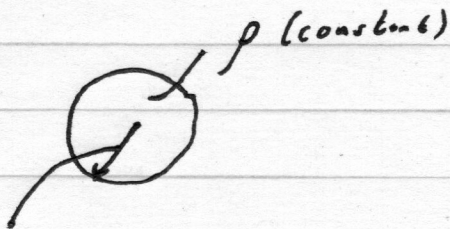
$$= \pi + \frac{1}{4} \int_0^{2\pi} \cos u du$$

$$= \pi + \frac{1}{4} \sin u \Big|_0^{2\pi}$$

Therefore Stokes' Theorem is true

$$\boxed{\int_S (\text{curl } \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \pi}$$

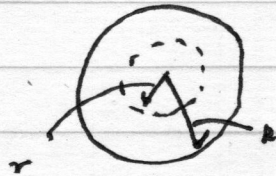
Problem 7



$$\oint_S \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0}$$

R

For $r < R$

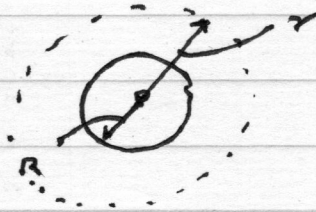


$$\begin{aligned} \oint_S \vec{E} \cdot d\vec{S} &= \oint_S E dS = E 4\pi r^2 \\ &= \int \frac{\rho dV}{\epsilon_0} = \frac{\rho}{\epsilon_0} \frac{4}{3} \pi r^3 \end{aligned}$$

$$E = \frac{\rho r}{3\epsilon_0}$$

$$\vec{E} = \frac{\rho r}{3\epsilon_0} \hat{r} \quad 0 \leq r \leq R$$

For $r > R$



$$\begin{aligned} \oint_S \vec{E} \cdot d\vec{S} &= \oint_S E dS = \\ &E 4\pi r^2 \end{aligned}$$

$$= \frac{1}{\epsilon_0} \rho \frac{4}{3} \pi R^3$$

$$\vec{E} = \frac{\rho R^3}{3\epsilon_0 r^2} \hat{r} \quad r \geq R$$

The First Maxwell Equation says

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

For $0 \leq r \leq R$

$$\vec{E} = \frac{\rho r \hat{r}}{3\epsilon_0}$$

From previous work in this course we know

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{1}{r^2} \frac{\partial (r^2 E_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (E_\theta \sin \theta)}{\partial \theta} \\ &\quad + \frac{1}{r \sin \theta} \frac{\partial E_\phi}{\partial \phi} \end{aligned}$$

in spherical polar coordinates

Fortunately we only have E_r so

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[\frac{r^2 \rho r}{3\epsilon_0} \right] = \frac{1}{r^2} \frac{\rho}{3\epsilon_0} 3r^2$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho = \frac{\rho}{\epsilon_0}$$

which makes sense since

$\rho \neq 0$ $0 \leq r \leq R$

For $r \geq R$

$$\vec{E} = \frac{\rho R^3 \hat{r}}{3\epsilon_0 r^2}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[\frac{r^2 \rho R^3}{3\epsilon_0 r^2} \right]$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left[\frac{\rho R^3}{3\epsilon_0} \right] = 0$$

which makes sense since

$$\rho = 0 \quad r \geq R$$

Problem 8

Where are charges?

$$3q \text{ at } a\hat{k}$$

$$q \text{ at } -a\hat{k}$$

$$-2q \text{ at } +j\hat{a}$$

$$-2q \text{ at } -j\hat{a}$$

Net charge is $3q + q - 2q - 2q = 0$ so there is no monopole moment.

$$V_{\text{dip}} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

What is \vec{p} ?

$$\begin{aligned} \vec{p} &= \int \vec{r}' \rho(\vec{r}') dV' = 3q a \hat{k} - a q \hat{k} \\ &\quad + (-2qa) \hat{j} + (2qa) \hat{j} \\ &= 2aq \hat{k} \end{aligned}$$

$$V_{\text{dip}} = V_{\text{dip}}(r, \theta) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} 2aq \hat{k} \cdot \hat{r}$$

$$\begin{aligned} \text{Since } \hat{r} &= \hat{i} \sin \theta \cos \phi + \hat{j} \sin \theta \sin \phi + \hat{k} \cos \theta \\ \hat{k} \cdot \hat{r} &= \cos \theta \end{aligned}$$

$$V_{\text{dip}}(r, \theta) = \frac{2aq \cos \theta}{4\pi\epsilon_0 r^2}$$

Problem 9

(a) total charge = $3q - q = 2q$

monopole moment = $2q$

(i) monopole potential = $\frac{2q}{4\pi\epsilon_0 r}$

(ii) dipole moment

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') dV' = ?$$

Charge at $a\hat{k}$ is $3q$

Charge at 0 is $-q$

$$\vec{p} = 3qa\hat{k}$$

(iii) potential at large r including both the monopole and dipole contributions

$$\frac{2q}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0 r^2} 3qa\hat{k} \cdot \hat{r}$$

Again using the results for $\hat{k} \cdot \hat{r}$ from previous problem

$$V = V(r, \theta) = \frac{2q}{4\pi\epsilon_0 r} + \frac{3qa \cos \theta}{4\pi\epsilon_0 r^2}$$

Problem 9

(i) total charge = $3q - q = 2q$

(ii) monopole moment = $2q$

monopole potential = $\frac{2q}{4\pi\epsilon_0 r}$

(iii) dipole moment

Charge of $3q$ at O

Charge of $-q$ at $-ka$

$\vec{p} = +qka$

(iii) potential at large r including both the monopole and dipole contributions

$$\frac{2q}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0 r^2} qa \hat{k} \cdot \hat{r}$$

or

$$V = V(r, \theta) = \frac{2q}{4\pi\epsilon_0 r} + \frac{qa \cos \theta}{4\pi\epsilon_0 r^2}$$

Problem 9

(c) total charge = $3q - q = 2q$

(i) magnetic moment = $2q$

magnetic potential = $\frac{2q}{4\pi\epsilon_0 r}$

(ii) dipole moment

Charge of $-q$ at O

Charge of $3q$ at $+j\hat{a}$

$$\vec{p} = 3qa\hat{j}$$

(iii) potential at large r including both the monopole and dipole contributions

$$\frac{2q}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0 r^2} 3qa \hat{j} \cdot \hat{r}$$

Recall

$$\hat{j} = \sin\theta \sin\phi \hat{r} + \cos\theta \sin\phi \hat{\theta} + \cos\phi \hat{\phi}$$

$$V = V(r, \theta) = \frac{2q}{4\pi\epsilon_0 r} + \frac{3qa \sin\theta \sin\phi}{4\pi\epsilon_0 r^2}$$