USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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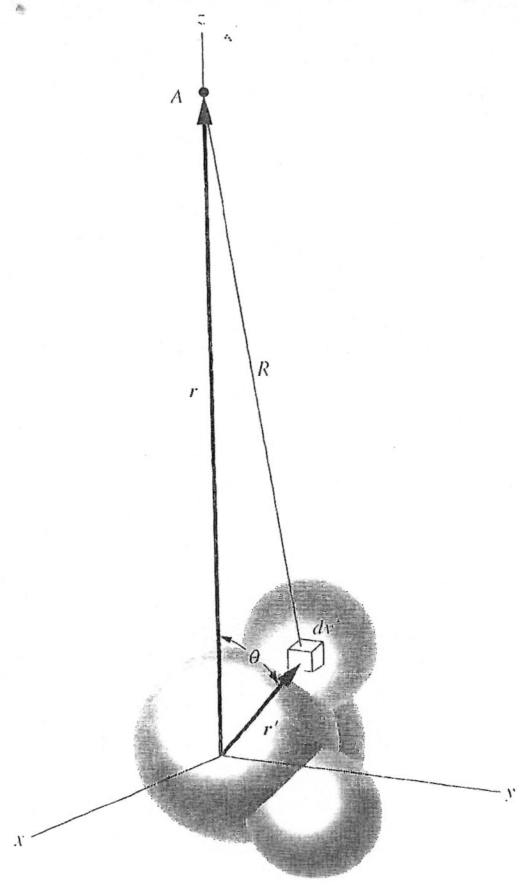
PROBLEM SET XI
(due Tuesday, September 8, 2020)

Problem 1

In Lecture 11 we discussed the electrostatic potential V due to a dipole moment $\vec{p}$ of two charges $+q$ and $-q$ separated by a distance d in which we made a number of approximations. Let us revisit this problem by looking at the electrostatic potential V

(1) $$V(A) = \frac{1}{4\pi\epsilon_o}\int_V \frac{\rho(x', y', z')\, dV'}{R}$$

of any arbitrary charge distribution (e.g. a molecular charge distribution) as shown below



In this problem we wish to find the electrostatic potential V at some point A along the z-axis. Since we are not assuming any special symmetry in the charge distribution, there is nothing special about the z-axis. In this figure we use primed coordinates to describe the charge distribution. Using the law of cosines we can define R as

(2) $$R = (r^2 + (r')^2 - 2\,r\,r'\,\cos\theta)^{\frac{1}{2}}$$

or

(3) $$(r^2 + (r')^2 - 2\,r\,r'\,\cos\theta)^{-\frac{1}{2}} = \frac{1}{r}\left[1 + \left(\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta\right)\right]^{-\frac{1}{2}}$$

If we next use a Taylor series (actually a Maclaurin series) expansion for the bracketed term show that we obtain

(4) $$(r^2 + (r')^2 - 2\,r\,r'\,\cos\theta)^{-\frac{1}{2}} = \frac{1}{r}\left[1 + \frac{r'}{r}\cos\theta + \left(\frac{r'}{r}\right)^2 \frac{(3\cos^2\theta - 1)}{2} + \cdots\right]$$

If we substitute (4) into (1) we obtain the following expression

(5) $$V(A) = \frac{1}{4\pi\epsilon_o}\left[\frac{K_0}{r} + \frac{K_1}{r^2} + \frac{K_2}{r^3} + \cdots\right]$$

where

(6) $$K_0 = \int \rho\; dV'$$

(7) $$K_1 = \int r'\,\cos\theta\;\rho\; dV'$$

(8) $$K_2 = \int r'^2\,\frac{(3\cos^2\theta - 1)}{2}\;\rho\; dV'$$

This power series is called the multipole expansion of the electrostatic potential (or just potential), although we have calculated it only for a point on the z-axis. To finish the problem we would have to get V at all other points in order to find the electric field as $-\nabla V$. We have gone far enough though to bring out the essential point that the behavior of the potential at large distances from the source will be dominated by the first term in the series of (5) which does not vanish.

Now show for our example of a dipole in Lecture 11 that the first non-vanishing term of (5) is

$$(9) V_{dipole}(A) = V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_o r^2}\int \hat{r}\cdot\vec{r}\,'\rho\, dV' = \frac{\hat{r}}{4\pi\epsilon_o r^2}\cdot\int \vec{r}\,'\rho\, dV' = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_o r^2}$$

where we define the dipole moment $\vec{p}$ as

$$(10) \qquad \vec{p} = \int \vec{r}\,'\rho\, dV'$$

and we can finally express the dipole potential as

$$(11) \qquad V_{dipole}(\vec{r}) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_o r^2}$$

or

$$(12) \qquad V_{dipole}(\vec{r}) = \frac{p\cos\theta}{4\pi\epsilon_o r^2}$$

where $\vec{p}$ points in the +z-direction. Note that our angle θ here has a different meaning from the angle θ previously used in this problem where it indicated the position of a point in the charge distribution. The present θ indicates the position of a given point at which we want to calculate $V(\vec{r})$ with respect to the dipole direction. We will come back to this idea in Problem 5!

Problem 2

Revisit the derivation in Lecture 11 where we showed that the dipole potential V of a system of two charges $+q$ and $-q$ separated by a distance d was given by

$$V_{dipole}(\vec{r}) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_o r^2} = \frac{\hat{r}\cdot(q\,\vec{d})}{4\pi\epsilon_o r^2}$$

and complete the missing steps from our derivation in lecture.

Problem 3

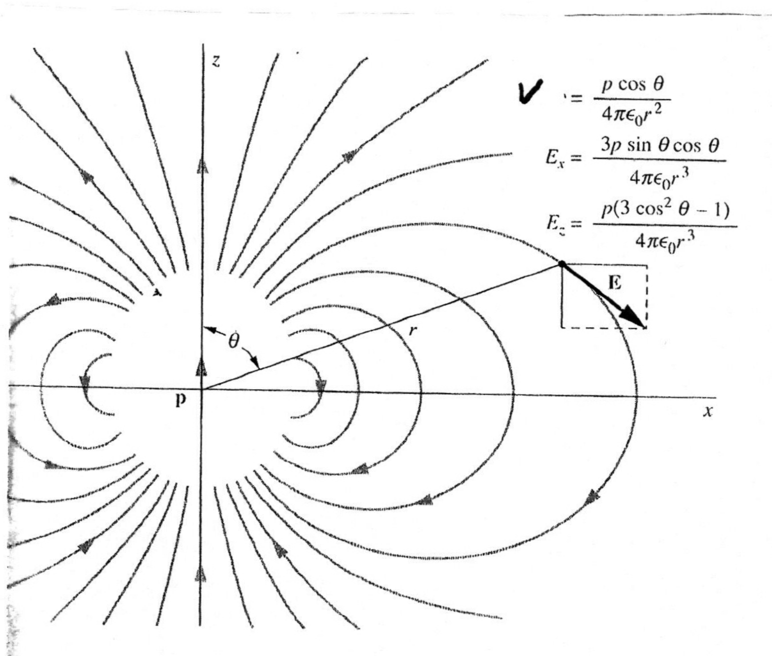
Using the dipole potential V_{dipole} in Problem 2 show that the electric field $\vec{E}$ is given by

$$E_r = \frac{p}{2\pi\epsilon_o r^3}\cos\theta$$

$$E_\theta = \frac{p}{4\pi\epsilon_o r^3}\sin\theta$$

$$E_\phi = 0$$

Verify that these vector components for the dipole for $\vec{E}$ in Cartesian coordinates are as listed in the plot below



Problem 4

Show that the electric field $\vec{E}$ of the dipole in Problem 3 can be written in the coordinate-free form

$$E_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_o}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

Hint: Recall our expressions for $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$, in terms of and $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ from Problem Set I.

Problem 5

A circular ring in the xy-plane of radius R centered at the origin carries a uniform linear charge density λ. Using the results in Problem 1 find the first three terms monopole ($n = 0$), dipole ($n = 1$), and quadrupole ($n = 2$) for the multipole expansion for $V(r, \theta)$. Note that you should probably tackle Problem 6 and Problem 7 first before this one. Up to now we have been using the multipole expansion in Problem 1 to find the electrostatic potential $V(r)$ at a point of high symmetry (e.g. along a particular path in space such as the z-axis or along the position vector emanating from the origin). Actually for that matter we have only been computing $V(r)$ for all the examples done so far in this course.

Now we have going to do something different in this problem as we wish to find $V(r, \theta)$ which is a more general problem! You simply need to remember that you now have $\vec{r}$ which depends on r, θ, ϕ and $\vec{r}'$ which depends on ρ', ϕ' in this problem. Once you have expressions for these two vectors, you will need to take their dot product to find the angle between them, which is what you are really going to need in this problem and not θ, as shown in Problem 1.

Problem 6

A sphere of radius R which is centered at the origin carries a volume charge density

$$\rho(r, \theta) = \frac{kR}{r^2}(R - 2r)\sin\theta$$

where k is a constant and r and θ are the usual spherical polar coordinates. Find the approximate electrostatic potential for points on the z-axis far away from the sphere.

Problem 7

Return to Problem 2 and expand the problem out to order $\left(\frac{d}{r}\right)^3$ and use this to find the quadrupole ($n = 2$) term and the octopole term ($n = 3$) in the potential.