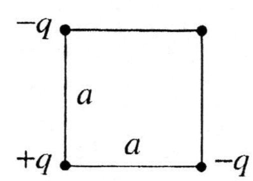
USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
Dr. Steven L. Richardson (srichards22@comcast.net)
Professor Emeritus of Electrical Engineering, Department of Electrical and Computer Engineering, Howard University, Washington, DC
and
Faculty Associate in Applied Physics, John A. Paulson School of Engineering and Applied Sciences, Harvard University, Cambridge, MA

PROBLEM SET X
(due September 1 2020)

Problem 1

Three charges are situated at the corners of a square of side a as shown in the figure below. How much work does it take to bring another charge $+q$ from far away and place it in the fourth corner?



Problem 2

Refer to the system previously discussed in Problem 1. How much work does it take to assemble the whole configuration of four charges?

Problem 3

Consider an infinite one-dimensional chain of point charges q and $-q$ with alternating signs strung out along the x-axis each with a distance a from its nearest neighbor. Compute the work per particle to assemble this system. As a hint consider the Maclaurin series expansion of

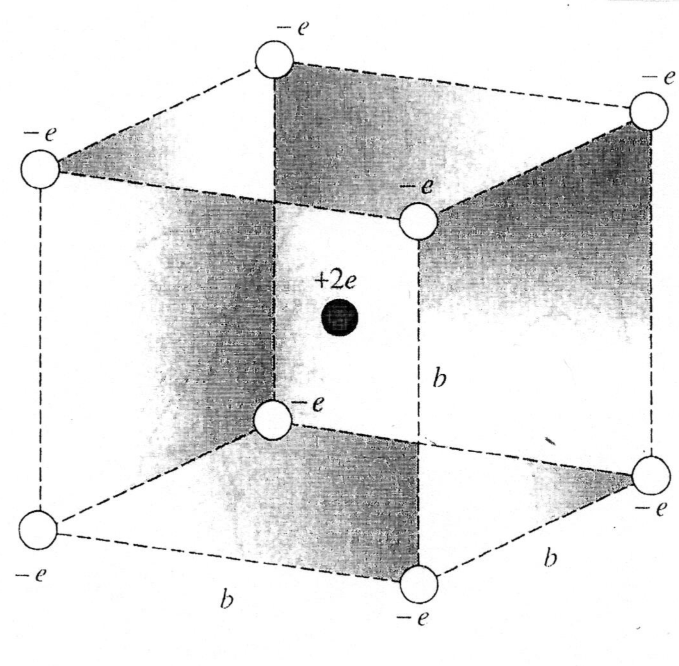
$$\ln(1+x)$$

Problem 4

Two positive charges point charges q_a and q_b of masses m_a and m_b respectively are at rest held together by a massless string of length a. If the string is cut, the two particles fly off in opposite directions. How fast is each one going when they are far apart?

Problem 5

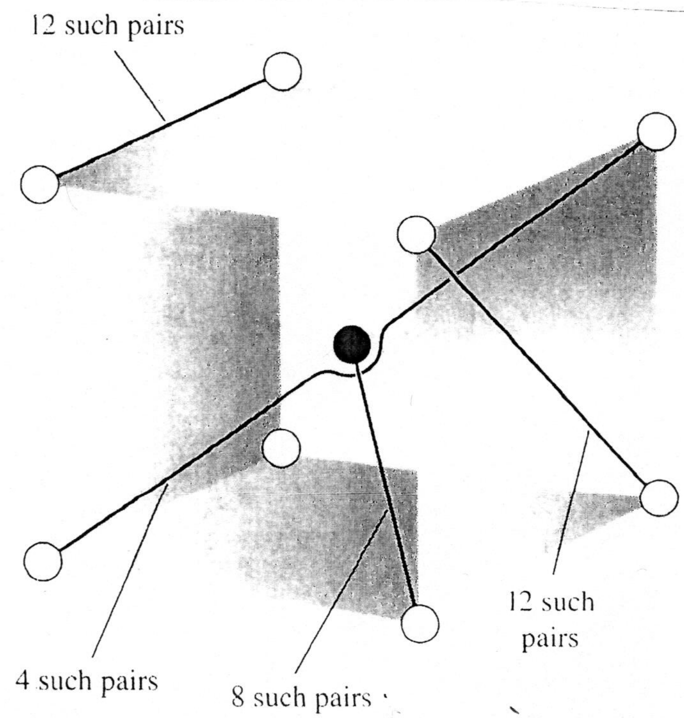
Show that the electrostatic potential energy U of an arrangement of eight separate negative charges $-q$ on the corners of a cube of side b with a positive charge $+2q$ located at the center of the cube.



is given by

$$U = \frac{1}{4\pi\epsilon_o}\left[\frac{-16\,q^2}{\frac{\sqrt{3}}{2}\,b} + \frac{12\,q^2}{b} + \frac{12\,q^2}{\sqrt{2}\,b} + \frac{4\,q^2}{\sqrt{3}\,b}\right] = \frac{q^2}{4\,b\,\pi\,\epsilon_o}\,4.31947$$

You can use as a hint the figure below where you should recall that we only have to worry about calculating pairs of interactions among charges when evaluating the electrostatic potential energy of a system.

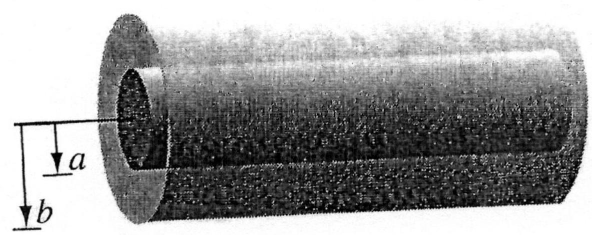


Problem 6

Find the capacitance of two coaxial spherical metal shells with radii r_a and r_b where $r_a < r_b$. Note that the inner sphere has a charge $+q$ while the outer sphere has a charge $-q$.

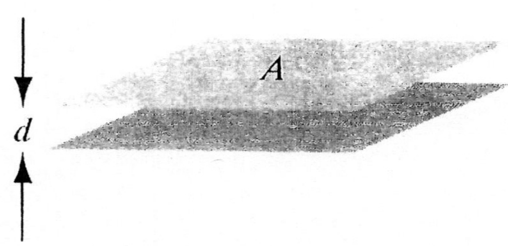
Problem 7

Find the capacitance per unit length of two coaxial metal cylindrical tubes of radii r_a and r_b where $r_a < r_b$ as shown in the figure below. Note that the inner cylinder has a surface charge density $+\sigma$ while the outer cylinder has a surface charge density $-\sigma$.



Problem 8

Consider the parallel-plate capacitor shown in the figure below



If you wish to "charge up" the capacitor, you have to remove electrons from the positive plate and move them to the negative plate. In doing so, you fight against the electric field which is pulling them back towards the positive plate and pushing them away from the negative one. How much work is then done to charge the capacitor up to some final amount Q? We can calculate this by supposing that at some intermediate stage the charge on the positive plate is q. This would mean that the electrostatic potential difference between the two plates of the capacitor is $\frac{q}{C}$. The work required to move the next piece of charge dq is therefore

$$dW = \frac{q}{C}\, dq$$

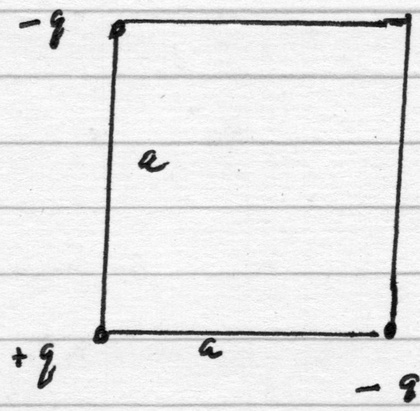
Show that the total work required to go from $q = 0$ to $q = Q$ is

$$W = \frac{1}{2} C V^2$$

where V is the final electrostatic potential of the capacitor.

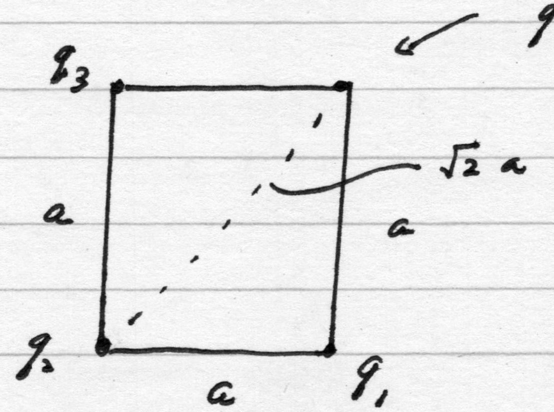
Problem 1

Three charges are situated at the corners of square of side a, as shown in the figure below. How much work does it take to bring in another charge $+q$ from far away and place it in the fourth center?



From Lecture 10 we know the extra work required to bring in q_4 to our system of charges is

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left[\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}} \right]$$



Using our figure we see

$$q_1 = -q \qquad r_{14} = a$$
$$q_2 = +q \qquad r_{24} = \sqrt{2}\, a$$
$$q_3 = -q \qquad r_{34} = a$$
$$q_4 = +q$$

$$W_4 = \frac{1}{4\pi\epsilon_0}\, q \left[\frac{-q}{a} + \frac{q}{\sqrt{2}\, a} + -\frac{q}{a} \right]$$

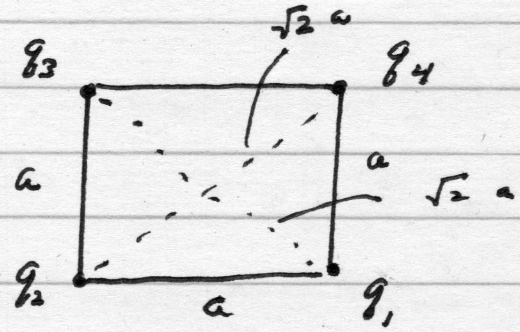
$$W_4 = \frac{q^2}{4\pi\epsilon_0 a} \left[-1 + \frac{1}{\sqrt{2}} - 1 \right]$$

$$\boxed{W_4 = \frac{q^2}{4\pi\epsilon_0 a} \left[\frac{1}{\sqrt{2}} - 2 \right]}$$

$W_4 < 0$ so system is stable

Problem 2

Refer to the system described in Problem 1. How much work does it take to assemble the whole configuration of four charges?



$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{4} \sum_{j\neq i}^{4} \frac{q_i q_j}{r_{ij}} =$$

$$\frac{1}{8\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_1}{r_{12}} + \frac{q_2 q_3}{r_{23}} \right.$$

$$+ \frac{q_2 q_4}{r_{34}} + \frac{q_3 q_1}{r_{31}} + \frac{q_3 q_2}{r_{32}} + \frac{q_3 q_4}{r_{34}}$$

$$\left. + \frac{q_4 q_1}{r_{41}} + \frac{q_4 q_2}{r_{42}} + \frac{q_4 q_3}{r_{43}} \right]$$

From our diagram

$q_1 = -q$ $\qquad$ $r_{12} = a$

$q_2 = +q$ $\qquad$ $r_{13} = \sqrt{2}\,a$

$q_3 = -q$ $\qquad$ $r_{14} = a$

$q_4 = +q$

$r_{21} = a$

$r_{31} = \sqrt{2}\,a$ $\qquad$ $r_{23} = a$

$r_{32} = a$ $\qquad$ $r_{24} = \sqrt{2}\,a$

$r_{34} = a$

$r_{41} = a$

$r_{42} = \sqrt{2}\,a$

$r_{43} = a$

$$W = \frac{1}{8\pi\epsilon_0}\left[\frac{-q^2}{a} + \frac{q^2}{\sqrt{2}\,a} \; \frac{-q^2}{a}\right.$$

$$\frac{-q^2}{a} \quad \frac{-q^2}{a} \quad \frac{+q^2}{\sqrt{2}\,a} \quad \frac{+q^2}{\sqrt{2}\,a} \quad \frac{-q^2}{a} \quad \frac{-q^2}{a}$$

$$\left.\frac{-q^2}{a} \quad \frac{+q^2}{\sqrt{2}\,a} \quad \frac{-q^2}{a}\right]$$

$$W = \frac{1}{8\pi\epsilon_0}\left[\frac{-8q^2}{a} + \frac{4q^2}{\sqrt{2}\,a}\right]$$

$$W = \frac{q^2}{2\pi\epsilon_0 a}\left[\frac{1}{\sqrt{2}} - 2\right] < 0 \quad \text{System is stable!}$$

Problem 3

Consider an infinite one-dimensional chain of point charges q and $-q$ with alternating signs strung out along the x-axis each with a distance a from its nearest neighbor. Compute the work per particle to assemble this system. As a hint consider the Maclaurin series expansion of

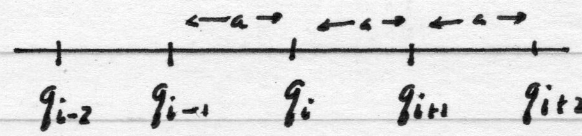
$$\ln(1+x)$$

This is a classic problem but it needs to be approached carefully as it has some subtle points that should not be overlooked.

Let us start with

$$(3-1) \qquad W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{q_i q_j}{r_{ij}}$$

which is the work required to assemble a system of N charges. In our problem we do not have N charges but an infinite number.
Let us pick one charge q_i in (3-1) and construct our system accordingly



If we select q_i then (3-1) can be expanded as

(3-2) $$W = \frac{1}{8\pi\epsilon_0}\left[\frac{q_i q_{i+1}}{a} + \frac{q_i q_{i-1}}{a} + \frac{q_i q_{i+2}}{2a} + \frac{q_i q_{i-2}}{2a} + \frac{q_i q_{i+3}}{3a} + \frac{q_i q_{i-3}}{3a} + \cdots\right]$$

Since $q_i = +q$ [by choice but all of the following still holds if we chose $q_i = -q$]

$q_{i+1} = -q$

$q_{i-1} = -q$

$q_{i+2} = +q$, etc

and (3-2) becomes

(3-3) $$W = \frac{1}{8\pi\epsilon_0}\left[-\frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{2a} + \frac{q^2}{2a} - \frac{q^2}{3a} - \frac{q^2}{3a} + \cdots\right]$$

or

(3-4) $$W = \frac{q^2}{4\pi\epsilon_0 a}\left[-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \cdots\right]$$

By factoring out -1 we get

(3-5) $$W = \frac{-q^2}{4\pi\epsilon_0 w}\left[1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots\right]$$

The terms in the bracket can be summed exactly if we use our hint and do a Maclaurin series expansion for $\ln(1+x)$

(3-6) $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

This series converges for $|x| < 1$, but we need to explore what happens at the radii of convergence, namely $x = 1$ and $x = -1$.

For $x = -1$ this series becomes the harmonic series which diverges

(3-7) $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}(-1)}{n}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n}$$

For $x = 1$ we find

$$(3\text{-}8) \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n)} x^n \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

which is the alternating harmonic series which converges conditionally.

Thus

$$(3\text{-}9) \qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

converges for x in the interval $-1 < x \leq 1$ and

$$(3\text{-}10) \qquad W = \frac{-q^2}{4\pi\epsilon_0 a} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (1)^n = \frac{-q^2}{4\pi\epsilon_0 a} \ln(1+1)$$

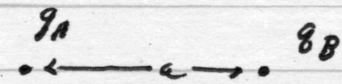
or

$$\boxed{(3\text{-}11) \qquad W = \frac{-q^2}{4\pi\epsilon_0 a} \ln 2}$$

Note that this is the work per particle required to assemble the infinite system of point charges in one-dimension

Problem 4

Two positive charges (point charges) of charges q_A and q_B of masses m_A and m_B respectively are at rest held together by a massless string of length a. If the string is cut, the two particles fly off in opposite directions. How fast is each one going when they are far apart?



Our system initially has no kinetic energy but only electrostatic potential energy U

$$U = \frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{a} \qquad (4\text{-}1)$$

When the string is cut the two particles fly off in opposite directions due to Coulomb's Law. At some far apart distance all of the initial electrostatic potential energy of the system is converted to the total kinetic energy of both particles, at which point V_A and V_B are constants.

$$\frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{a} = \frac{m_A V_A^2}{2} + \frac{m_B V_B^2}{2} \qquad (4\text{-}2)$$

We need to find V_A, V_B but we lack a second equation. Conservation of momentum comes to our rescue

$$(4\text{-}3) \qquad m_A V_A = m_B V_B$$

where V_A and V_B are the velocities of the two charges when they are far apart. If we combine (4-2) and (4-3) we discover

$$\frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{a} = \frac{m_A}{2} V_A^2 + \frac{m_B}{2}\left(\frac{m_A}{m_B}\right)^2 V_A^2$$

or

$$V_A^2\left[\frac{m_A}{2} + \frac{m_A^2}{2m_B}\right] = \frac{q_A q_B}{4\pi\epsilon_0 a}$$

or

$$V_A^2\left[\frac{m_A m_B}{2m_B} + \frac{m_A^2}{2m_B}\right] = \frac{q_A q_B}{4\pi\epsilon_0 a}$$

$$V_A^2\left[\left(\frac{m_A}{m_B}\right)\left[\frac{m_B}{2} + \frac{m_A}{2}\right]\right] = \frac{q_A q_B}{4\pi\epsilon_0 a}$$

$$V_A = \left(\frac{q_A q_B}{4\pi\epsilon_0 a}\right)^{1/2}\left[\left(\frac{m_A}{m_B}\right)\left[\frac{m_A + m_B}{2}\right]\right]^{-1/2}$$

$$V_A = \left(\frac{q_A q_B}{4\pi\epsilon_0 a} \right)^{1/2} \left[\left(\frac{m_A}{m_B} \right) \left[\frac{m_A + m_B}{2} \right] \right]^{-1/2}$$

$$V_A = \left[\frac{q_A q_B}{4\pi\epsilon_0 a} \right]^{1/2} \left[\frac{m_B}{m_A} \right]^{1/2} \left[\frac{m_A + m_B}{2} \right]^{-1/2}$$

$$V_A = \left[\frac{q_A q_B}{4\pi\epsilon_0 a} \right]^{1/2} \left[\left(\frac{m_B}{m_A} \right) \frac{2}{m_A + m_B} \right]^{1/2}$$

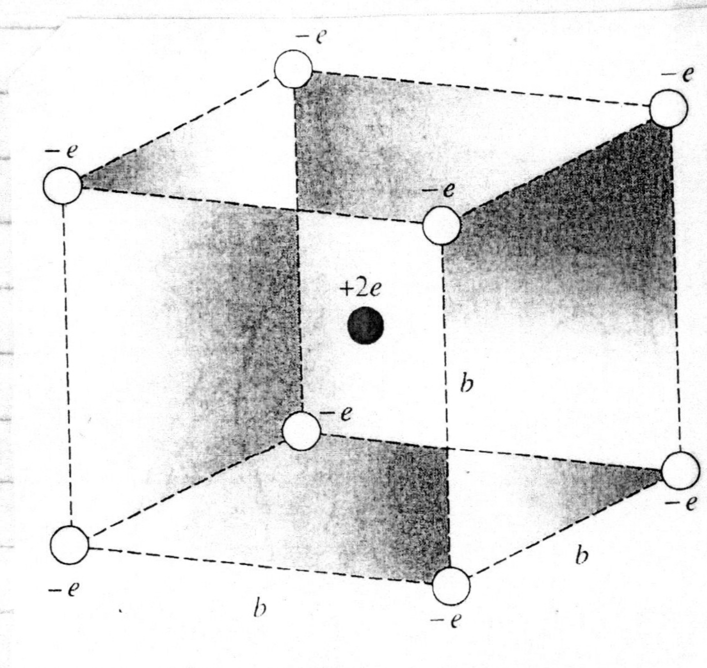
$$\boxed{V_A = \left[\frac{q_A q_B}{2\pi\epsilon_0 a} \left(\frac{m_B}{m_A} \right) \frac{1}{m_A + m_B} \right]^{1/2}} \qquad (4\text{-}4)$$

Similarly

$$\boxed{V_B = \left[\frac{q_A q_B}{2\pi\epsilon_0 a} \left(\frac{m_A}{m_B} \right) \frac{1}{m_A + m_B} \right]^{1/2}} \qquad (4\text{-}5)$$

Problem 5

Show that the electrostatic potential energy U of an arrangement of eight separate negative charges $-q$ on the corners of a cube of side b with a positive charge of $+2q$ located at the center of the cube



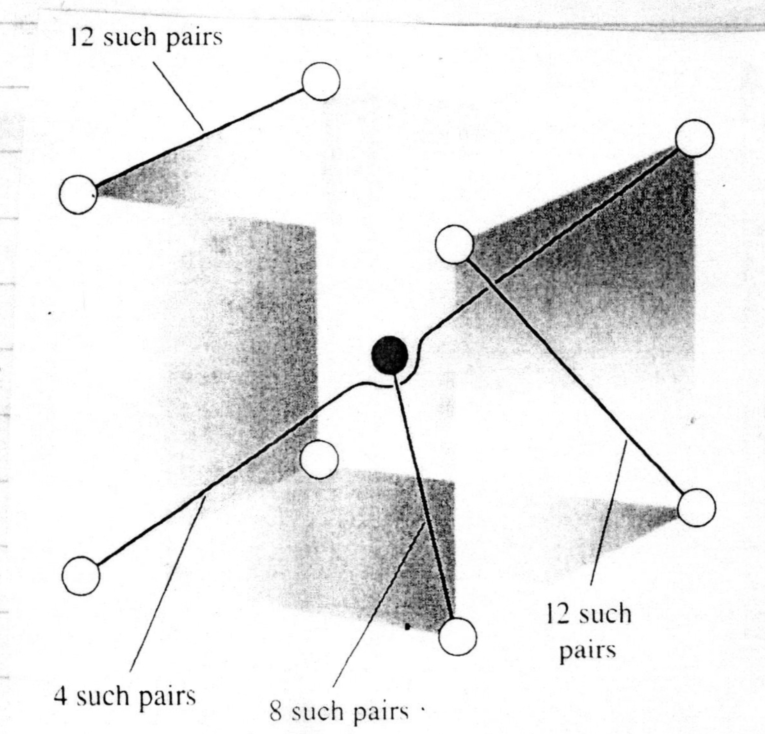
is given by

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{-16q^2}{\frac{\sqrt{3}}{2}b} + \frac{12q^2}{b} + \frac{12q^2}{\sqrt{2}b} + \frac{4q^2}{\sqrt{3}b}\right]$$

or

$$U = \frac{q^2}{4b\pi\epsilon_0}\ 4.31947\ldots$$

You can use as a hint the figure below where you should recall we only need worry about calculating pairs of interactions among charges when evaluating the electrostatic potential energy of a system.

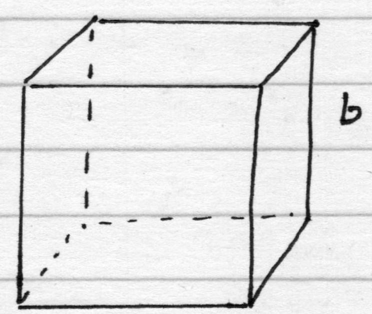


Given

$$U = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j\neq i}^{N} \frac{q_i q_j}{r_{ij}}$$

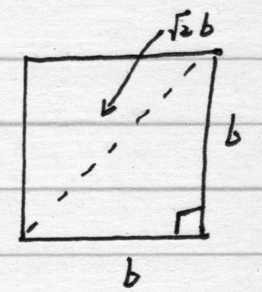
we only need consider pairs of interactions among charges once so we don't need our factor of 1/2

$$U = \frac{1}{4\pi\epsilon_0} \left[\text{\# pairs of type 1} + \text{\# pairs of type 2} + \cdots \right]$$



For our cube:

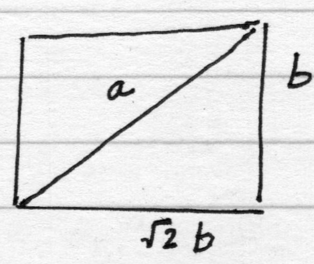
(a) Face diagonal is $\sqrt{2}\, b$



since

$b^2 + b^2 = 2b^2$

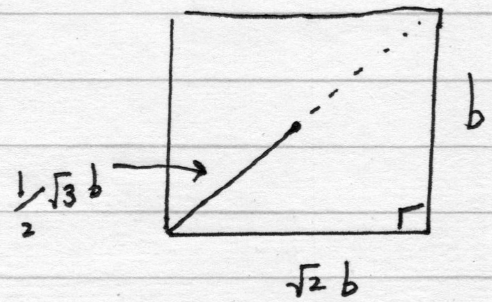
(b) Cube diagonal



$a^2 = 2b^2 + b^2$

$a = \sqrt{3}\, b$

(c) Half of cube diagonal



From our figure:

There are 12 pairs of interactions between $-q$ and $-q$ of seperation b

There are six cube faces and 12 pairs of interactions between $-q$ and $-q$ considering face diagonal interactions of distance $\sqrt{2}\,b$

There are four pairs of interactions between $-q$ and $-q$ considering body diagonal interactions of distance $\sqrt{3}\,b$

There are eight pairs of interactions between $-q$ and $+2q$ considering half the distance $\frac{\sqrt{3}b}{2}$ for body diagonal interactions

If we put all of this together, we obtain

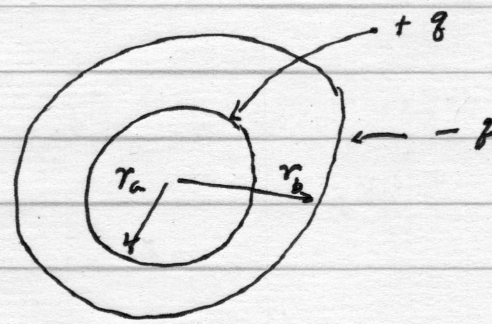
$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{8(-2q^2)}{\frac{\sqrt{3}}{2}b} + \frac{12q^2}{b} + \frac{12q^2}{\sqrt{2}b} + \frac{4q^2}{\sqrt{3}b}\right]$$

$$U = \frac{q^2}{4b\pi\epsilon_0}(14.31947) > 0$$

You have to do work to assemble this system as it is unstable and it will fly apart left to its own devices (sans external force)!

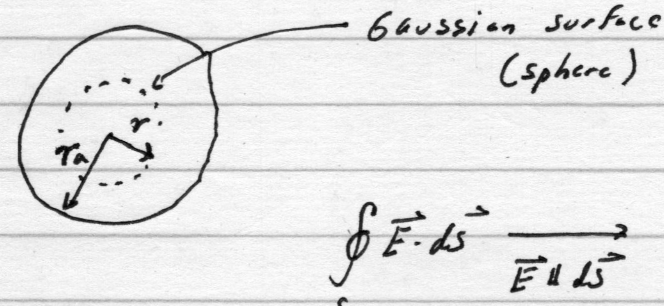
Problem 6

Find the capacitance of two coaxial spherical metal shells with radii r_a and r_b where $r_a < r_b$. Note that the inner sphere has a charge $+q$ while the outer sphere has a charge $-q$.



Use Gauss's Law in each of the three regions

Region I $(0 < r < r_a)$

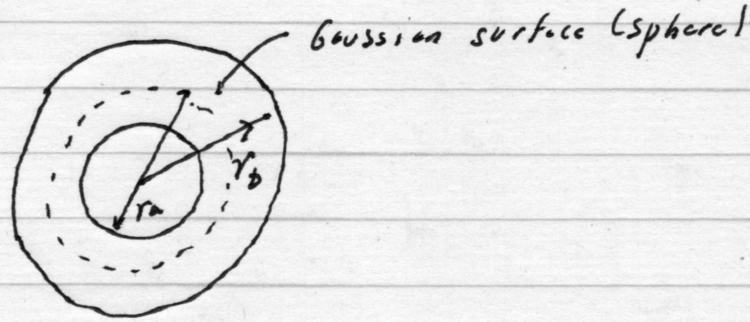


$$\oint_S \vec{E}\cdot d\vec{S} \xrightarrow[\text{by symmetry}]{\vec{E} \parallel d\vec{S}} \oint_S \vec{E}\cdot d\vec{S} \rightarrow \oint_S E\,dS$$

$\downarrow$ E must be constant over S

Thus $\vec{E} = 0$ $\qquad 0 = \dfrac{q_{enc}}{\epsilon_0} \longleftarrow E\,4\pi r^2 \longleftarrow E \oint_S dS$

Region II $(r_a < r < r_b)$

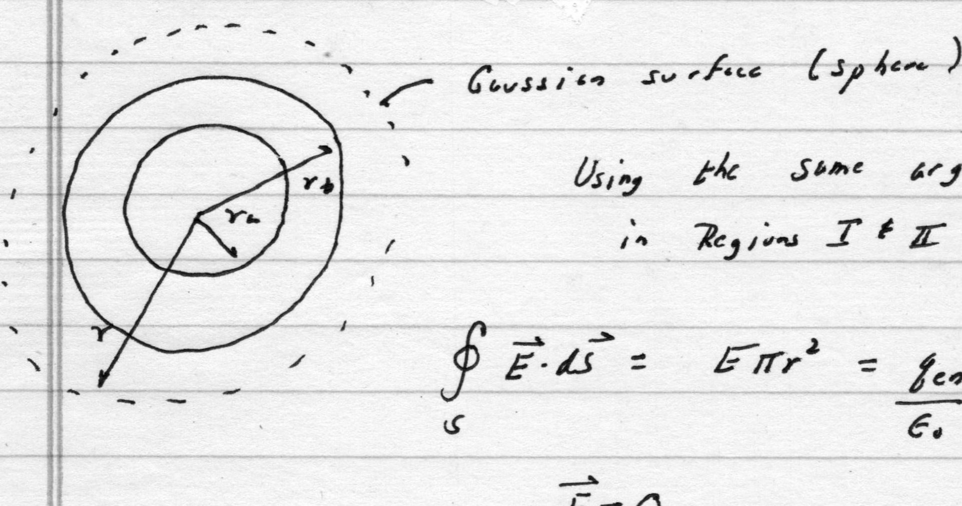


Using the same arguments as in Region I

$$\oint_S \vec{E}\cdot d\vec{S} = E4\pi r^2 = \frac{q_{en}}{\epsilon_0} = \frac{1}{\epsilon_0} + q$$

$$\vec{E} = \frac{q}{4\pi r^2 \epsilon_0}\hat{r}$$

Region III $(r_b < r < \infty)$



Using the same arguments as in Regions I & II

$$\oint_S \vec{E}\cdot d\vec{S} = E\pi r^2 = \frac{q_{en}}{\epsilon_0} = \frac{[+q-q]}{\epsilon_0} = 0$$

$$\vec{E} = 0$$

There is only a $\vec{E}$ field in Region II !

$$V = -\int_{\infty}^{0} \vec{E}\cdot d\vec{r} = \overbrace{-\int_{\infty}^{r_b} \cancel{\vec{E}\cdot d\vec{r}}}^{0} \overbrace{-\int_{r_b}^{r_a} \cancel{\vec{E}\cdot d\vec{r}}}^{\text{NOT ZERO}} \overbrace{-\int_{r_a}^{0} \cancel{\vec{E}\cdot d\vec{r}}}^{0}$$

$$V = -\int_{r_b}^{r_a} \frac{q}{4\pi r^2 \epsilon_0}\, dr$$

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right] > 0$$

since $r_b > r_a$

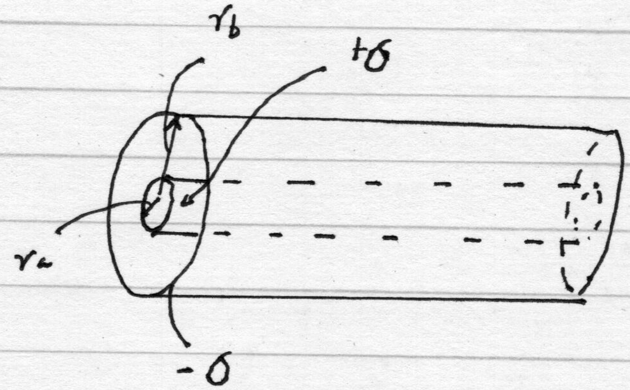
$$C = \frac{q}{V} = \frac{q}{V} = q\left[\frac{q}{4\pi\epsilon_0}\left(\frac{r_b - r_a}{r_a r_b}\right)\right]^{-1}$$

$$C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a} > 0$$

This is entirely a geometric factor!

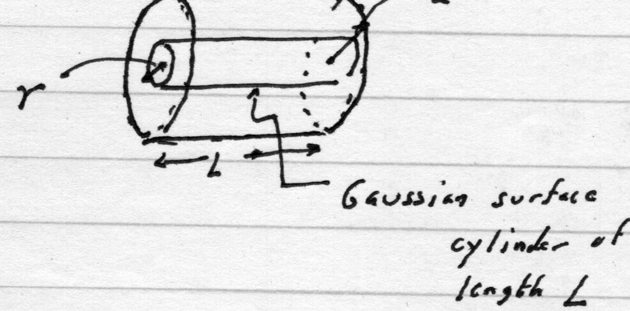
Problem 7

Find the capacitance per unit length of two coaxial metal tubes of radii r_a and r_b where $r_a < r_b$ as shown in the figure below. Note that the inner cylinder has a surface charge density $+\sigma$ while the outer cylinder has a surface charge density $-\sigma$.



Use Gauss's Law in each of the three regions

Region I $\quad 0 < r < r_a$



Break Gaussian surface up into 3 sections:

$$\oint_S \vec{E}\cdot d\vec{s} = \int_{TOP} \vec{E}\cdot d\vec{s} + \int_{BOTTOM} \vec{E}\cdot d\vec{s} + \int_{SIDE} \vec{E}\cdot d\vec{s}$$

By symmetry $\vec{E} \parallel \vec{dS}$ along side
and $\vec{E} \perp \vec{dS}$ on top and bottom
so only surface integral over side
of Gaussian cylinder survives

$$\oint_S \vec{E}\cdot\vec{dS} = \int_{SIDE} E\,dS = E\int dS = E\,2\pi r L$$

E is constant over S by symmetry

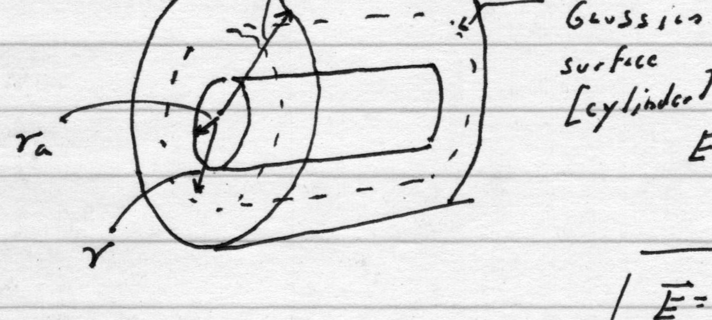
$$= \frac{q_{en}}{\epsilon_0} = 0$$

$$\vec{E} = 0$$

Region II $\quad r_a < r < r_b$

Using the same arguments as discussed in Region I

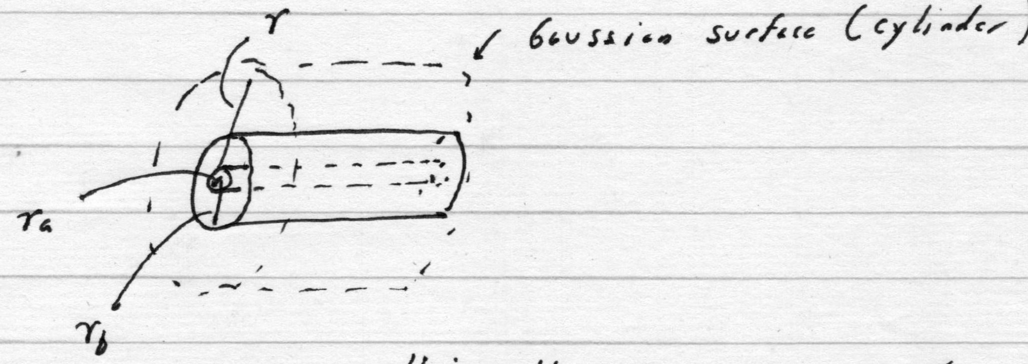
$$\oint_S \vec{E}\cdot\vec{dS} = E\,2\pi r L = \frac{q_{en}}{\epsilon_0} = \frac{\sigma}{\epsilon_0}A = \frac{\sigma\, 2\pi r_a L}{\epsilon_0}$$



$$E = \frac{\sigma}{\epsilon_0}\frac{r_a}{r}$$

$$\boxed{\vec{E} = \frac{\sigma}{\epsilon_0}\frac{r_a}{r}\hat{\rho}}$$

<u>Region III</u> $\quad r_b < r < \infty$



Using the same arguments as discussed in Region I and Region II

TOTAL ENCLOSED SURFACE CHARGE DENSITY

$$\oint_S \vec{E}\cdot d\vec{s} = E\,2\pi r = \frac{\sigma A}{\epsilon_0} = \frac{A}{\epsilon_0}\left[\sigma - \sigma\right] = 0$$

$$\vec{E} = 0$$

To find the capacitance we must compute the potential difference

$$V = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r} = -\int_{\infty}^{r_b} \cancel{\vec{E}\cdot d\vec{r}}^{\,0} - \int_{r_b}^{r_a} \vec{E}\cdot d\vec{r} - \int_{r_a}^{r} \cancel{\vec{E}\cdot d\vec{r}}^{\,0}$$

$$= -\int_{r_b}^{r_a} \frac{\sigma}{\epsilon_0}\frac{r_a}{r}\,dr = -\frac{\sigma}{\epsilon_0} r_a \ln\left(\frac{r_a}{r_b}\right)$$

$$V = \frac{\sigma}{\epsilon_0} r_a \ln\left(\frac{r_b}{r_a}\right) > 0$$

$$C = \frac{q}{V} = \frac{\epsilon_0}{\sigma\, r_a} \frac{1}{\ln\left(\frac{r_b}{r_a}\right)} q$$

$$C = \frac{q\epsilon_0}{r_a} \frac{1}{\ln\left(\frac{r_b}{r_a}\right)} \left(\frac{q}{A}\right)^{-1}$$

$$C = \frac{q\epsilon_0}{r_a} \frac{2\pi r_a L}{q} \frac{1}{\ln\left(\frac{r_b}{r_a}\right)}$$

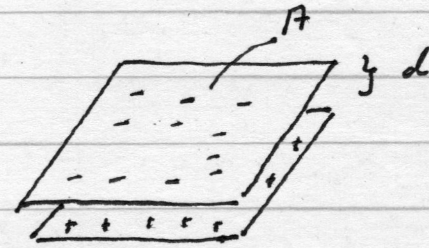
$$C = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{r_b}{r_a}\right)}$$

$$\boxed{\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\left(\frac{r_b}{r_a}\right)}}$$

Capacitance per unit length

Problem 8

Consider the parallel-plate capacitor shown in the figure below



If you wish to "charge up" the capacitor, you have to remove electrons from the positive plate and move them to the negative plate. In doing so, you fight against the electric field which is pulling them back towards the positive plate and pushing them away from the negative one. How much work is then done to charge the capacitor up to some final amount Q? We can calculate this by supposing that at some intermediate state the charge on the positive plate is q. This would mean that the electrostatic potential difference between the two plates of the capacitor is

$$V = \frac{q}{C}$$

The work required to move the next piece of charge dq is therefore

$$dW = V\,dq = \frac{q}{C}\,dq$$

Show that the total work required to go from $q=0$ to $q=Q$ is

$$W = \frac{1}{2} CV^2$$

$$W = \int dW = \int V\,dq = \int_0^Q \frac{q}{C}\,dq = \frac{Q^2}{2C}$$

Since V is the final electrostatic potential of the capacitor

$$C = \frac{Q}{V}$$

and

$$W = \frac{Q^2}{2C} = \frac{C^2V^2}{2C} = \frac{1}{2} CV^2$$

or

$$\boxed{W = \frac{1}{2} CV^2}$$