USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
Dr. Steven L. Richardson (srichards22@comcast.net)
Professor Emeritus of Electrical Engineering, Department of Electrical and Computer Engineering, Howard University, Washington, DC
and
Faculty Associate in Applied Physics, John A. Paulson School of Engineering and Applied Sciences, Harvard University, Cambridge, MA

PROBLEM SET VIII
(due Tuesday, August 18, 2020)

Problem 1

In Lecture 8 we found the electrostatic potential V and electric field $\vec{E}$ at a distance z above the center of the charge distribution of two identical charges q separated by a distance d. Repeat this calculation for the case where we changed the right-hand charge from q to $-q$. What then is the electrostatic potential V at the point $\vec{P}$? What field does that suggest? Compare your answer to Problem 5 in Problem Set 5 and carefully explain any discrepancy.

Problem 2

In Lecture 8 we found the electrostatic potential V and electric field $\vec{E}$ at a distance z above the origin of a finite line charge of length $2L$ and uniform linear charge density λ. Repeat this calculation with the integration steps we skipped in lecture.

Problem 3

In Lecture 8 we found the electrostatic potential V and electric field $\vec{E}$ everywhere for a spherical shell of radius a. Repeat this calculation with the integration steps we skipped in lecture.

Problem 4

Repeat Problem 2 for the case where the wire is infinite.

Problem 5

Using the following equation for the electrostatic potential $V(r)$

$$V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r'}$$

find the electrostatic potential V and electric field $\vec{E}$ everywhere for a uniformly charged solid sphere of radius a and whose total charge is q. Use infinity as your reference point. Check that your results are in agreement with the results we previously obtained in this course for this problem. Sketch V.

Problem 6

Given the system discussed in Problem 4 of Problem Set VII, find the electrostatic potential V at the center using infinity as your reference point. Use the following equation for the electrostatic potential $V(r)$

$$V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r'}$$

Problem 7

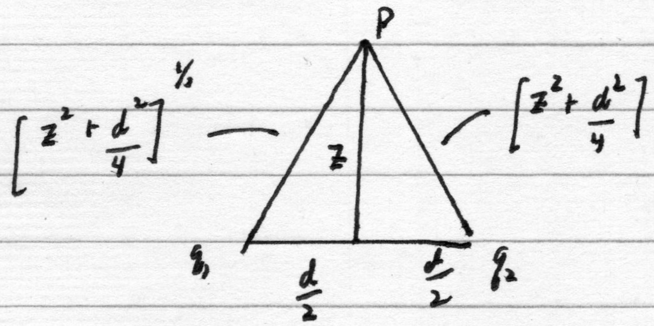
Given the system discussed in Problem 7 of Problem Set VII, find the electrostatic potential difference between a point on the axis and a point on the outer cylinder. Note that it is not necessary to commit yourself to a particular reference point if you use the appropriate equation. Start with the following equation for the electrostatic potential $V(r)$

$$V(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r'}$$

Problem 1

In Lecture 8 we found that the electrostatic potential V and electric field $\vec{E}$ at a distance z above the center of the charge distribution of two identical charges q separated by a distance d. Repeat this calculation for the case where we changed the right-hand charge from q to $-q$.

We previously discovered for the case



$q_1 = q_2 = +q$

$$V_1 = \frac{q_1}{4\pi\epsilon_0 |\vec{r}-\vec{r}_1{}'|} = \frac{q_1}{4\pi\epsilon_0 \left[z^2+\frac{d^2}{4}\right]^{1/2}}$$

$$V_2 = \frac{q_2}{4\pi\epsilon_0 |\vec{r}-\vec{r}_2{}'|} = \frac{q_2}{4\pi\epsilon_0 \left[z^2+\frac{d^2}{4}\right]^{1/2}}$$

Now the principle of superposition tells us that at $\vec{P}$

$$V = V_1 + V_2 = \frac{q_1}{4\pi\epsilon_0\left[z^2+\frac{d^2}{4}\right]^{1/2}} + \frac{q_2}{4\pi\epsilon_0\left[z^2+\frac{d^2}{4}\right]^{1/2}}$$

and since $q_1 = q_2 = +q$

$$V = \frac{2q}{4\pi\epsilon_0\left[z^2+\frac{d^2}{4}\right]^{1/2}}$$

and

$$\vec{E} = -\vec{\nabla} V = \frac{\hat{k}\, 2q\, z}{4\pi\epsilon_0 \left[z^2 + \frac{d^2}{4}\right]^{3/2}}$$

Now in this case let us replace q_2 by $-q$ so

$$V_1 = \frac{q}{4\pi\epsilon_0 \left[z^2 + \frac{d^2}{4}\right]^{1/2}}$$

$$V_2 = \frac{-q}{4\pi\epsilon_0 \left[z^2 + \frac{d^2}{4}\right]^{1/2}}$$

and using the principle of superposition at $\vec{P}$. <u>What then is the electrostatic potential V at the point $\vec{P}$?</u>

$$V = V_1 + V_2 = 0.$$

<u>What field does this suggest?</u>

Since $\vec{E} = -\vec{\nabla} V$

this would suggest that $\vec{E} = 0$ at $\vec{P}$ which contradicts what we know previously that

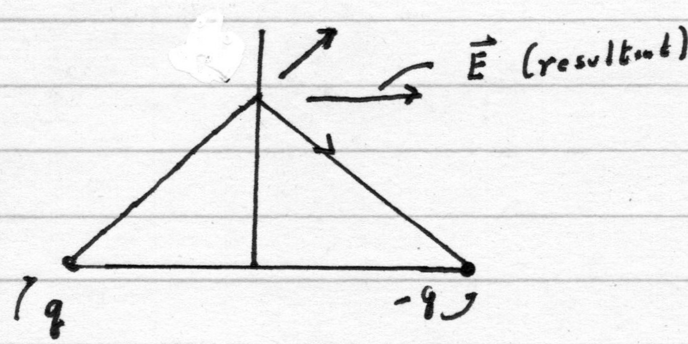
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{z} \hat{k}$$

What is the resolution to this dilemma?

The point is that we only know V on the z-axis and not everywhere! We can not hope to find

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y}.$$

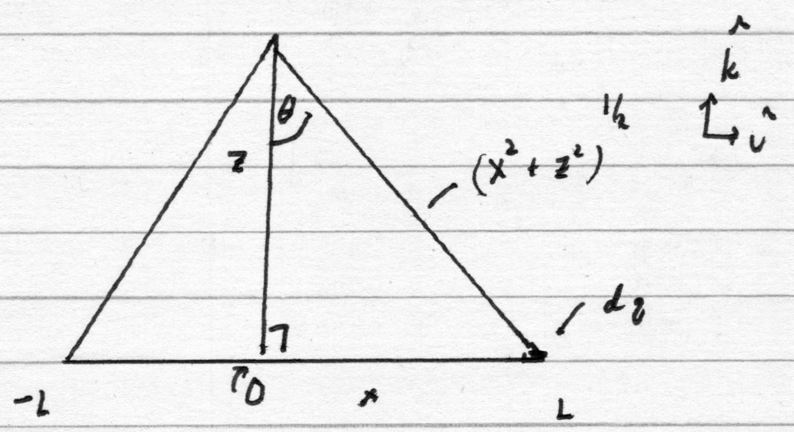
In our previous problem this was fine as from symmetry we know that $E_x = E_y = 0$, but now $\vec{E}$ points in the $+x$ direction



We can not possibly find $\vec{E}$ from $\vec{\nabla} V$ since only knowing V on the z-axis is insufficient to find $\vec{E}$. We need a knowledge of V everywhere in space which this problem does not give us. This, indeed, is a challenging problem to think about!

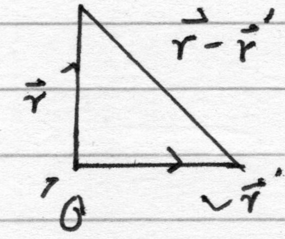
Problem 2

In Lecture 8 we found that the electrostatic potential V and electric field $\vec{E}$ at a distance z above the origin of a finite line charge of length $2L$ and uniform linear charge density λ. Repeat this calculation with the integration steps we skipped in lecture.



$$V = \int \frac{dq}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|} = \lambda \int \frac{dx}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

What is $|\vec{r}-\vec{r}'|$?



From above figure

$$|\vec{r}-\vec{r}'| = (x^2+z^2)^{1/2}$$

Thus

$$V(z) = \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{L} \frac{dx}{[x^2+z^2]^{1/2}}$$

where x is a variable and z is fixed

Let

$$[x^2+z^2]^{1/2} = z\left[1+\left(\frac{x}{z}\right)^2\right]^{1/2}$$

$$\tan\theta = \frac{x}{z}$$

$$x = z\tan\theta$$

$$dx = z\sec^2\theta\, d\theta$$

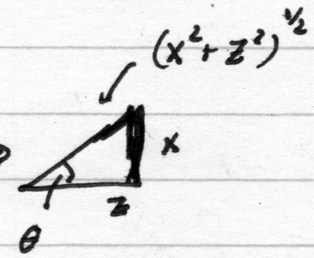
$$[x^2+z^2]^{1/2} = z[1+\tan^2\theta]^{1/2} = z\sec\theta$$

$$V = \frac{\lambda}{4\pi\epsilon_0}\int\frac{z\sec^2\theta\, d\theta}{z\sec\theta} = \frac{\lambda}{4\pi\epsilon_0}\int\sec\theta\, d\theta$$

We have done this integral before in this course!

$$V = \frac{\lambda}{4\pi\epsilon_0}\ln[\tan\theta+\sec\theta]$$

$\tan\theta = \frac{x}{z} \Rightarrow$



$$\cos\theta = \frac{z}{(x^2+z^2)^{1/2}}$$

$$\sin\theta = \frac{x}{(x^2+z^2)^{1/2}}$$

$$V = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{x}{z} + \frac{(x^2+z^2)^{1/2}}{z} \right] \Bigg|_{-L}^{L}$$

$$V(z) = \frac{\lambda}{4\pi\epsilon_0} \left[\ln\left(\frac{L}{z} + \frac{(L^2+z^2)^{1/2}}{z} \right) \right.$$

$$\left. - \ln\left(\frac{-L}{z} + \frac{(L^2+z^2)^{1/2}}{z} \right) \right]$$

$$V(z) = \frac{\lambda}{4\pi\epsilon_0} \left[\ln \left\{ \frac{(L + (L^2+z^2)^{1/2})}{(-L + (L^2+z^2)^{1/2})} \right\} \right.$$

or

$$\boxed{V(z) = \frac{\lambda}{4\pi\epsilon_0} \ln \left[\frac{(L^2+z^2)^{1/2} + L}{(L^2+z^2)^{1/2} - L} \right]}$$

Now let us find $\vec{E} = -\vec{\nabla} V$

$$E_z \approx -\frac{\partial}{\partial z}\left[\ln \alpha\right] \approx -\frac{1}{\alpha}\frac{d\alpha}{dz}$$

$$\alpha = \frac{(L^2+z^2)^{1/2} + L}{(L^2+z^2)^{1/2} - L}$$

$$\frac{d\alpha}{dz} = \frac{1}{\left[(L^2+z^2)^{1/2} - L\right]^2}$$

$$\times \left[\left[(L^2+z^2)^{1/2} - L\right] \frac{1}{2} (L^2+z^2)^{-1/2} \, 2z \right.$$

$$\left. - \left[(L^2+z^2)^{1/2} + L\right] \frac{1}{2} (L^2+z^2)^{-1/2} \, 2z \right]$$

$$\frac{d\alpha}{dz} = \frac{z}{\left[(L^2+z^2)^{1/2} - L\right]^2} \times$$

$$\left[\frac{(L^2+z^2)^{1/2} - L - L - (L^2+z^2)^{1/2}}{(L^2+z^2)^{1/2}} \right]$$

$$\frac{d\alpha}{dz} = \frac{-2Lz}{\left[(L^2+z^2)^{1/2} - L\right]^2} \; \frac{1}{(L^2+z^2)^{1/2}}$$

$$\alpha = \frac{(L^2+z^2)^{1/2} + L}{(L^2+z^2)^{1/2} - L}$$

$$\frac{1}{\alpha} = \frac{(L^2+z^2)^{1/2} - L}{(L^2+z^2)^{1/2} + L}$$

$$\frac{1}{\alpha}\frac{d\alpha}{dz} = \frac{-2Lz}{(L^2+z^2)^{1/2}} \frac{1}{\left[(L^2+z^2)^{1/2}+L\right]\left[(L^2+z^2)^{1/2}-L\right]}$$

$$\frac{1}{\alpha}\frac{d\alpha}{dz} = \frac{-2Lz}{(L^2+z^2)^{1/2}} \frac{1}{(L^2+z^2)-L^2}$$

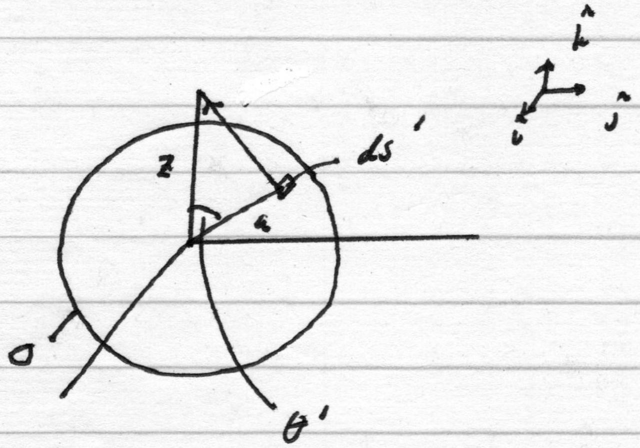
$$\frac{1}{\alpha}\frac{d\alpha}{dz} = \frac{-2Lz}{z^2(L^2+z^2)^{1/2}} = \frac{-2L}{z(L^2+z^2)^{1/2}}$$

$$E_z = \frac{\lambda}{4\pi\epsilon_0}\left(-\frac{1}{\alpha}\frac{d\alpha}{dz}\right)$$

$$\boxed{E_z = \frac{2\lambda L}{4\pi\epsilon_0 z} \frac{1}{[L^2+z^2]^{1/2}}}$$

Problem 3

In Lecture 8 we found the electrostatic potential V and electric field $\vec{E}$ everywhere for a spherical shell of radius a. Repeat this calculation with the integration steps we skipped in lecture.



The electrostatic potential V is given by

$$(3\text{-}1) \qquad V(\vec{r}) = \int \frac{dq}{4\pi\epsilon_0 |\vec{r} - \vec{r}\,'|}$$

Where

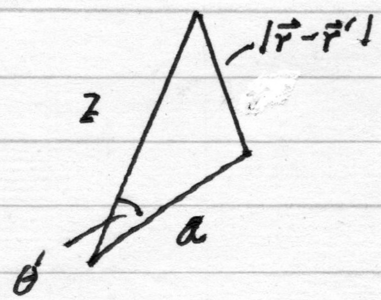
$$(3\text{-}2) \qquad dq = \sigma\, dS' \qquad \text{[differential charge]}$$

It is useful to designate primed coordinates when referring to our charge distribution. In spherical polar coordinates

$$(3\text{-}3) \qquad dS' = a^2 \sin\theta'\, d\theta'\, d\phi'$$

where a is a constant.

We need to express $|\vec{r}-\vec{r}'|$ in terms of z, a, θ' by using the Law of Cosines



$$|\vec{r}-\vec{r}'|^2 = z^2 + a^2 - 2za\cos\theta' \qquad (3-4)$$

If we solve for $|\vec{r}-\vec{r}'|$

$$(3\text{-}5) \qquad |\vec{r}-\vec{r}'| = [z^2 + a^2 - 2az\cos\theta']^{\frac{1}{2}}$$

and (3-1) becomes

$$(3\text{-}6) \qquad V(z) = \frac{\sigma}{4\pi\epsilon_0}\int \frac{a^2 \sin\theta' \, d\theta' \, d\phi'}{[z^2 + a^2 - 2az\cos\theta']^{\frac{1}{2}}}$$

Where z is fixed and a is a constant. Our variables are θ', ϕ', the polar and azimuthal angles respectively.

Now how do we perform this surface integral?

$$(3\text{-}7) \qquad V(z) = \frac{\sigma a^2}{4\pi\epsilon_0}\int_S \frac{\sin\theta' \, d\theta' \, d\phi'}{[z^2 + a^2 - 2az\cos\theta']^{\frac{1}{2}}}$$

First let us integrate over ϕ'

(3-8) $$V(Z) = \frac{\sigma a^2 2\pi}{4\pi\epsilon_0} \int \frac{\sin\theta' d\theta'}{[Z^2 + a^2 - 2aZ\cos\theta']^{1/2}}$$

Now let us integrate by substitution

(3-9) $$U = Z^2 + a^2 - 2aZ\cos\theta'$$

(3-10) $$dU = +2aZ\sin\theta' d\theta'$$

Using (3-9) and (3-10), we can recast (3-8) as

(3-11) $$V(Z) = \frac{\sigma a^2}{2\epsilon_0} \frac{1}{2aZ} \int \frac{dU}{U^{1/2}}$$

or

(3-12) $$V(Z) = \frac{\sigma a}{4\epsilon_0 Z} \int U^{-1/2} dU$$

Since we are integrating over θ' we can use (3-9) to determine the appropriate limits of integration

θ'	U
0	$(Z-a)^2$
π	$(Z+a)^2$

Now we have

$$(3\text{-}13) \qquad V(Z) = \frac{\sigma a}{4\epsilon_0 Z} \int_{(Z-a)^2}^{(Z+a)^2} U^{-1/2}\, dU$$

or

$$(3\text{-}14) \qquad V(Z) = \frac{\sigma a}{4\epsilon_0 Z} \left. \frac{U^{1/2}}{(\frac{1}{2})} \right|_{(Z-a)^2}^{(Z+a)^2}$$

or

$$(3\text{-}15) \qquad V(Z) = \frac{\sigma a}{2\epsilon_0 Z} \left[\sqrt{(Z+a)^2} - \sqrt{(Z-a)^2} \right]$$

Now we always have a choice for taking the square roots in (3-15). Specifically if we take the positive square root for $\sqrt{(Z-a)^2}$ then

$$(3\text{-}16) \qquad \sqrt{(Z-a)^2} = +(Z-a)$$

and our result is only valid for $Z > a$

Using (3-16) we have for (3-15)

(3-17) $$V(z) = \frac{\sigma a}{2\epsilon_0 z}\left[(z+a) - (z-a)\right]$$

or

(3-18) $$V(z) = \frac{\sigma a}{2\epsilon_0 z}(2a) = \frac{\sigma a^2}{\epsilon_0 z}$$

which has the correct units. Now let us find $\vec{E}$

(3-19) $$\vec{\nabla} V = \left(\frac{\partial}{\partial z}\left[\frac{\sigma a^2}{\epsilon_0 z}\right]\right) = \frac{\sigma a^2}{\epsilon_0}\left(-\frac{1}{z^2}\right)$$

and

(3-20) $$\vec{E} = -\vec{\nabla} V = \frac{\sigma a^2}{\epsilon_0 z^2}$$

Recall that

(3-21) $$\sigma = \frac{q}{4\pi a^2}$$

so that (3-20) becomes

(3-22) $$\|\vec{E}\| = \frac{\sigma a^2}{\epsilon_0 z^2} = \frac{q}{4\pi a^2}\frac{a^2}{\epsilon_0 z^2} = \frac{q}{4\pi\epsilon_0 z^2}$$

or

(3-23) $$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} \qquad \text{for } z > a$$

which is the correct answer!

Now what happens when $z < a$?
In order to take the positive square root in (3-16) we must have

(3-24) $$\sqrt{(z-a)^2} = \sqrt{(a-z)^2} = +(a-z)$$

for $a > z$. In this case (3-15) becomes

(3-25) $$V(z) = \frac{\sigma a}{2\epsilon_0 z}\left[z + a - (a - z) \right]$$

or

(3-26) $$V(z) = \frac{\sigma a}{2\epsilon_0 z} 2z = \frac{\sigma a z}{\epsilon_0 z} = \frac{\sigma a}{\epsilon_0}$$

and $\vec{E}$ becomes

(3-27) $$\vec{E} = -\vec{\nabla} V = -\left(\frac{\partial}{\partial z}\left(\frac{\sigma a z}{\epsilon_0 z}\right)\right) = 0$$

or

(3-28)
$\vec{E}$ vanishes
for $z < a$

Problem 4

Repeat Problem 2 for the case where the wire is infinite.

All we need do here is grab the result from Problem 2 where

$$V(z) = \frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{(L^2+z^2)^{1/2}+L}{(L^2+z^2)^{1/2}-L}\right]$$

Let us simplify

$$(L^2+z^2)^{1/2} = L\left(1+\frac{z^2}{L^2}\right)^{1/2}$$

and consider the case where $z \gg L$ (or $L\to\infty$)

$$(L^2+z^2)^{1/2} \approx L$$

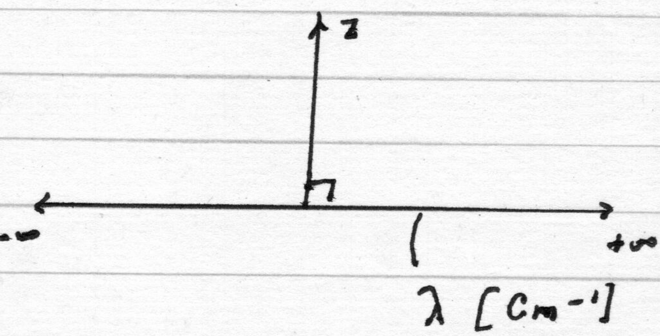
and

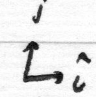
$$V(z) = \frac{\lambda}{4\pi\epsilon_0} \ln\left[\frac{L+L}{L-L}\right]$$

where we have a problem as $\ln\left[\frac{L+L}{L-L}\right]$ diverges! This is to be expected as the original form for the potential of a point charge is only valid if V vanishes when $r\to\infty$. This can not happen for an infinite system!

Let us tackle this problem using a different approach!

We start with our physical problem of an infinite straight wire of linear charge density λ





and we wish to evaluate the line integral

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}\,'$$

We know that

$$\vec{E} = \frac{\hat{k}\, 2\lambda}{4\pi\epsilon_0 z}$$

from previous work. We will only get a contribution to this line integral if we integral not over the path of the wire (because $\hat{k} \cdot \hat{i}\, dx = 0$) but over the path in the $+z$ direction. Thus

$$d\vec{r}\,' = d\vec{z}\,'$$

and $$\vec{E} \cdot d\vec{z}\,' = \frac{2\lambda}{4\pi\epsilon_0 z'} (\hat{k} \cdot \hat{k}\, dz') = \frac{2\lambda\, dz'}{4\pi\epsilon_0 z'}$$

Next we can not set $V=0$ at $r=\infty$ for an infinite charge distribution if we want to avoid a divergence in performing our integral.

Let us evaluate the line integral at some point a as a lower limit of integration

$$V = -\int_a^z \frac{2\lambda\, dz'}{4\pi\epsilon_0 z'} = \frac{-\lambda}{2\pi\epsilon_0} \ln z' \Big]_a^z$$

or

$$V(z) = \frac{-\lambda}{2\pi\epsilon_0} \ln\left(\frac{z}{a}\right)$$

where a is a constant.

Now we can find $\vec{E}$ as always

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}\left[\frac{-\lambda}{2\pi\epsilon_0}\ln z + \frac{\lambda}{2\pi\epsilon_0}\ln a\right]$$

$$\vec{E} = \frac{+\lambda}{2\pi\epsilon_0 z}\hat{k}$$ which makes sense!

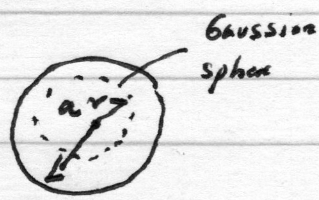
Problem 5

Using the following equation for the electrostatic potential $V(r)$

$$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r}\,'$$

find the electrostatic potential V and electric field $\vec{E}$ everywhere for a uniformly charged sphere of radius a and whose total charge is q. Use infinity as your reference point. Check that your results are in agreement with the results we previously obtained in this problem. Sketch V.

For $0 < r < a$



use Gauss's Law

$$\oint_S \vec{E}\cdot d\vec{s} \longrightarrow E\,4\pi r^2 = \frac{q_{en}}{\epsilon_0}$$

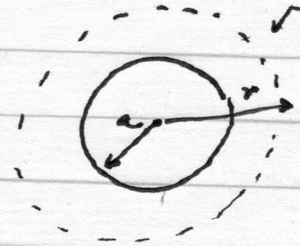
We have done this step so many times before

$$= \frac{\rho}{\epsilon_0}\frac{4}{3}\pi r^3$$

$$E = \frac{\rho\, r}{\epsilon_0 3} = \frac{q\,3\,r}{3\epsilon_0\, 4\pi a^3}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{r}{a^3}\hat{r}}$$

For $a < r < \infty$



$$\oint_S \vec{E}\cdot d\vec{S} \longrightarrow \qquad E\,4\pi r^2 = \frac{q_{en}}{\epsilon_0}$$

Again we have done this step so many times before

$$E\,4\pi r^2 = \frac{\rho}{\epsilon_0}\,\frac{4}{3}\pi a^3$$

$$\vec{E} = = \frac{\rho}{3\epsilon_0}\,\frac{a^3}{r^2}\,\hat{r} = \frac{q}{3\epsilon_0}\,\frac{3}{4\pi a^3}\,\frac{a^3}{r^2}\,\hat{r}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{r}}$$ as expected

For $r > a$

$$V = -\int_\infty^r \vec{E}\cdot d\vec{r}\,' = -\int_\infty^r \frac{q}{4\pi\epsilon_0}\,\frac{dr'}{(r')^2}$$

$$V = \frac{q}{4\pi\epsilon_0}\,\frac{1}{r'}\Big|_\infty^r$$

$$V(r) = \frac{q}{4\pi\epsilon_0}\frac{1}{r} - \frac{q}{4\pi\epsilon_0}\frac{1}{\infty}$$

$$\boxed{V(r) = \frac{q}{4\pi\epsilon_0 r}} \qquad r > a$$

For $\quad r < a$

$$V = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r}\,' = -\int_{\infty}^{a} \vec{E}\cdot d\vec{r}\,' - \int_{a}^{r} \vec{E}\cdot d\vec{r}\,'$$

$$V = -\int_{\infty}^{a} \frac{q}{4\pi\epsilon_0 (r')^2}\,\hat{r}\cdot d\vec{r}\,' - \int_{a}^{r} \frac{q}{4\pi\epsilon_0 a^3}\, r'\,\hat{r}\cdot d\vec{r}\,'$$

$$V = -\frac{q}{4\pi\epsilon_0}\int_{\infty}^{a} \frac{dr'}{(r')^2} - \frac{q}{4\pi\epsilon_0 a^3}\int_{a}^{r} r'\,dr'$$

$$V = -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r'}\right]_{\infty}^{a} - \frac{q}{4\pi\epsilon_0 a^3}\left.\frac{(r')^2}{2}\right|_{a}^{r}$$

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a} - \cancel{\frac{1}{\infty}}^{\,0}\right] - \frac{q}{4\pi\epsilon_0 a^3}\left[\frac{r^2}{2} - \frac{a^2}{2}\right]$$

$$V(r) = \frac{q}{4\pi\epsilon_0 a} - \frac{q r^2}{8\pi\epsilon_0 a^3} + \frac{a^2 q}{8\pi\epsilon_0 a^3}$$

$$V(r) = \frac{2q}{8\pi\epsilon_0 a} + \frac{q}{8\pi\epsilon_0 a} - \frac{qr^2}{8\pi\epsilon_0 a^3}$$

$$V(r) = \frac{3q}{8\pi\epsilon_0 a} - \frac{qr^2}{8\pi\epsilon_0 a^3}$$

$$V(r) = \frac{q}{8\pi\epsilon_0 a}\left[3 - \frac{r^2}{a^2}\right], \quad r < a$$

Confirm that these results make sense!

For $r > a$ $\quad V(r) = \dfrac{q}{4\pi\epsilon_0 r}$

$$E_r = -\frac{\partial}{\partial r}(V_r) = -\frac{\partial}{\partial r}\left(\frac{q}{4\pi\epsilon_0 r}\right) = \frac{q}{4\pi\epsilon_0 r^2}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} \quad \text{as expected}$$

For $r < a$

$$E_r = -\frac{\partial}{\partial r}(V_r) = -\frac{\partial}{\partial r}\left[\frac{q}{8\pi\epsilon_0 a}\left[3 - \frac{r^2}{a^2}\right]\right]$$

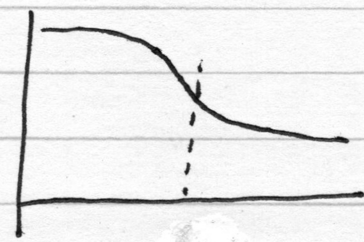
$$= -\frac{\partial}{\partial r}\left(\frac{-r^2}{a^2}\left[\frac{q}{8\pi\epsilon_0 a}\right]\right)$$

$$= +\frac{2r}{a^2}\frac{q}{8\pi\epsilon_0 a} = \frac{qr}{4\pi\epsilon_0 a^3}$$

$$\vec{E} = \frac{q r}{4\pi\epsilon_0 a^3}\,\hat{r} \quad , \text{ as expected}$$

Plot $V(r)$

$$\frac{V(r)}{\dfrac{q}{4\pi\epsilon_0 a}}$$



1 $\quad \dfrac{r}{a}$ $\quad \left[r \text{ in units of } a \right]$

$V(r)$ in units of $\dfrac{q}{4\pi\epsilon_0 a}$

Problem 6

Given the system discussed in Problem 4 of Problem set VII, find the electrostatic potential at the center using infinity as your reference point. Use the following equation for the electrostatic potential $V(r)$

$$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r}\,'$$

From Problem 4

$$\vec{E} = 0 \qquad 0 < r < a$$

$$\vec{E} = \frac{k}{\epsilon_0}\,\frac{(r-a)\,\hat{r}}{r^2} \qquad a < r < b$$

$$\vec{E} = \frac{k}{\epsilon_0 r^2}\,(b-a)\,\hat{r} \qquad r > b$$

$$V(r) = -\int_{\infty}^{0} \vec{E}\cdot d\vec{r} = -\int_{\infty}^{b} \vec{E}\cdot d\vec{r} - \int_{b}^{a} \vec{E}\cdot d\vec{r} - \int_{a}^{0} \vec{E}\cdot d\vec{r}$$

Clearly the last integral vanishes because

$\vec{E} = 0\ , \quad 0 < r < a$

Let us tackle the first integral

$$-\int_{\infty}^{b} \vec{E}\cdot d\vec{r} = -\int_{\infty}^{b} \frac{k}{\epsilon_0 r^2}(b-a)\,dr$$

$$= -\frac{k}{\epsilon_0}(b-a)\int_{\infty}^{b} r^{-2}\,dr$$

$$= \frac{-k}{\epsilon_0}(b-a)\left.\frac{r^{-2+1}}{-2+1}\right|_{\infty}^{b}$$

$$= \frac{+k}{\epsilon_0}(b-a)\left.\frac{1}{r}\right|_{\infty}^{b}$$

$$= \frac{k}{\epsilon_0}(b-a)\left[\frac{1}{b} - \cancelto{0}{\frac{1}{r}}\right]$$

$$= \frac{k}{\epsilon_0}\frac{(b-a)}{b} \quad \checkmark$$

Let us look at the second integral

$$-\int_b^a \vec{E}\cdot d\vec{r} = -\int_b^a \frac{k}{\epsilon_0} \frac{(r-a)\hat{r}}{r^2}\cdot d\vec{r}$$

$$= -\int_b^a \frac{k}{\epsilon_0} \frac{(r-a)}{r^2}\,dr$$

$$= -\frac{k}{\epsilon_0}\int_b^a \frac{r\,dr}{r^2} + \frac{k}{\epsilon_0}\int_b^a \frac{a\,dr}{r^2}$$

$$= \frac{-k}{\epsilon_0}\int_b^a \frac{dr}{r} + \frac{ka}{\epsilon_0} \left.\frac{r^{-2+1}}{-2+1}\right|_b^a$$

$$= \frac{-k}{\epsilon_0} \ln\left(\frac{a}{b}\right) - \frac{ka}{\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

If we combine everything

$$V(r) = \frac{k}{\epsilon_0}\frac{(b-a)}{b} - \frac{k}{\epsilon_0}\ln\left(\frac{a}{b}\right) - \frac{ka}{\epsilon_0}\frac{(b-a)}{ab}$$

$$V(r) = \frac{k}{\epsilon_0}\frac{(b-a)}{b} - \frac{k}{\epsilon_0}\ln\left(\frac{a}{b}\right) - \frac{k}{\epsilon_0}\frac{(b-a)}{b}$$

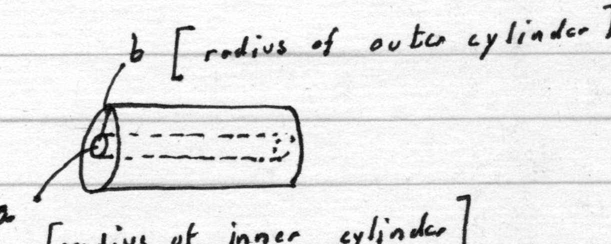
$$\boxed{V(r) = \frac{-k}{\epsilon_0}\ln\left(\frac{a}{b}\right)} \quad \checkmark$$

or $V(r) = \frac{k}{\epsilon_0}\ln\left(\frac{b}{a}\right)$

Problem 7

Given the system discussed in Problem 7 of Problem Set VII, find the electrostatic potential difference between a point on the axis and a point on the outer cylinder. Note that it is not necessary to commit yourself to a particular reference point if you use the appropriate equation. Use the following equation for the electrostatic potential $V(r)$

$$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{r}\,'$$



From Problem 7

$$\vec{E} = \frac{1}{2\epsilon_0}\rho\rho_0\hat{\rho} \quad , \rho < a$$

$$\vec{E} = \frac{\rho_0 a^2}{2\epsilon_0 \rho}\hat{\rho}, \qquad a<\rho<b$$

$$\vec{E} = 0 \qquad \rho > b$$

We wish to find

$$V(b) - V(0) = -\int_0^b \vec{E} \cdot d\vec{r}$$

$$V(b) - V(0) = -\int_0^a \vec{E} \cdot d\vec{r} - \int_a^b \vec{E} \cdot d\vec{r}$$

$$V(b) - V(0) = -\int_0^a \frac{1}{2\epsilon_0} \rho \rho_v \, d\rho - \int_a^b \frac{\rho_v a^2}{2\epsilon_0 \rho} \, d\rho$$

$$V(b) - V(0) = \frac{-\rho_v}{2\epsilon_0} \frac{\rho^2}{2} \Big|_0^a - \frac{\rho_v a^2}{2\epsilon_0} \ln \rho \Big|_a^b$$

$$V(b) - V(0) = \frac{-\rho_v a^2}{4\epsilon_0} - \frac{\rho_v a^2}{2\epsilon_0} \left[\ln b - \ln a \right]$$

$$V(b) - V(0) = \frac{-\rho_v a^2}{4\epsilon_0} - \frac{\rho_v a^2}{2\epsilon_0} \ln\left(\frac{b}{a}\right)$$

$$V(b) - V(0) = \frac{-\rho_v a^2}{4\epsilon_0} \left[1 + 2\ln\left(\frac{b}{a}\right) \right]$$