USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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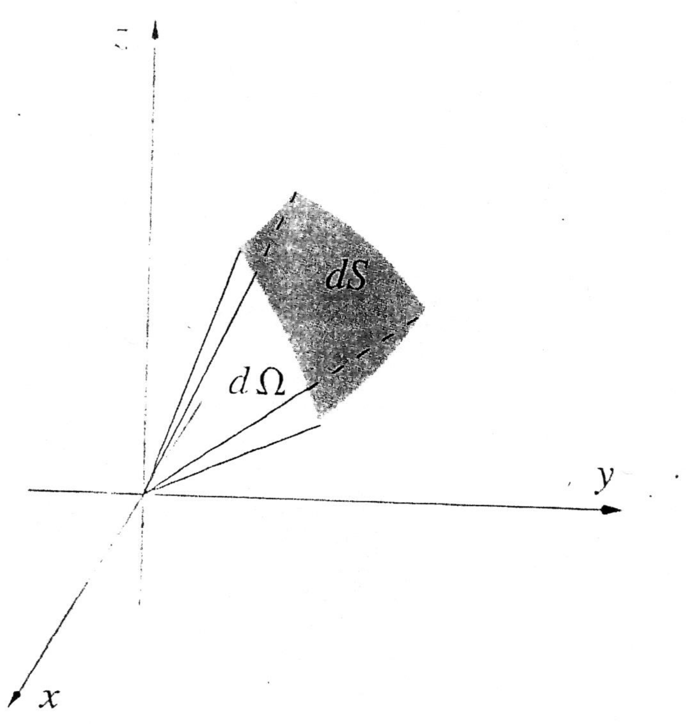
PROBLEM SET VII
(due Tuesday, August 11, 2020)

Problem 1

In Lecture 7 we proved Gauss's Law for a single point charge using a spherical Gaussian surface. We now need to show that Gauss's Law is true for any general closed surface S. Let us first review the idea of a solid angle. Let S be an area on a sphere of radius r centered on the origin. All the rays starting at the origin and passing through S form a cone, which is the solid angle Ω. We say that Ω is subtended by S. The units of solid angles are steradians, just as the units of planar angles are radians. The figure below shows the solid angle $d\Omega$ subtended by dS.



Just as the arc length ds on a circle is related to the angle $d\theta$ in radians that it subtends by $ds = r\,d\theta$ where r is the radius of the circle, dS is related to the solid angle $d\Omega$ (in steradians) that it subtends by $dS = r^2\,d\Omega$. For example, if $r = a =$ constant, then the total surface area of the sphere is $S = 4\pi a^2$ so that a complete solid angle is 4π, just as a complete angle for a circle is 2π.

Now let us apply this to the problem at hand. Consider the figure below for a single point charge centered by Gaussian spherical surface S and a general Gaussian surface S'. All of the appropriate items are defined in the figure below

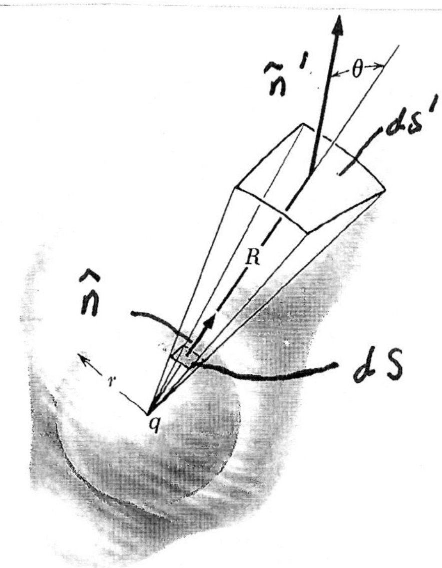


Fig. 1.16 Showing that the flux through *any* closed surface around q is the same as the flux through the sphere.

By carefully applying what we have discussed so far to this figure we realize that

$$\vec{E} = \frac{q\,\hat{r}}{4\pi\epsilon_o\,r^2} \tag{1}$$

$$\vec{E} = \frac{q\,\hat{r}}{4\pi\epsilon_o\,R^2} \tag{2}$$

$$dS = r^2\,d\Omega \tag{3}$$

$$dS' = R^2\, d\Omega \tag{4}$$

Now let us determine the surface integral or flux of $\vec{E}'$ through S'

$$\oint_{S'} \vec{E}' \cdot d\vec{S}' = \oint_{S'} E' dS'\, \hat{r} \cdot \hat{n}' = \oint_{S'} \frac{q}{4\pi\epsilon_o R^2}\, dS'\, \hat{r} \cdot \hat{n}' \tag{5}$$

and since the solid angle Ω subtended by dS' is the same as the solid angle subtended by dS

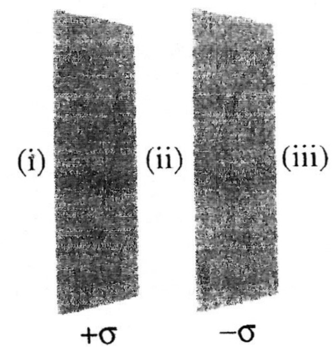
$$\oint_{S'} \vec{E}' \cdot d\vec{S}' = \oint_{S'} \frac{q}{4\pi\epsilon_o r^2} dS\, \hat{r} \cdot \hat{n}' = \oint_{S} \frac{q}{4\pi\epsilon_o r^2}\, dS\, \hat{r} \cdot \hat{n} \tag{6}$$

or where in the last surface integral we realize we are now integrating over S so that $\hat{n}'$ now becomes $\hat{n}$ to obtain the desired result

$$\oint_{S'} \vec{E}' \cdot d\vec{S}' = \oint_{S} \vec{E} \cdot d\vec{S} \tag{7}$$

Problem 2

Two infinite parallel planes carry equal but opposite uniform charge densities σ and $-\sigma$ as shown in the figure below.



Find the electric field in each of the three regions: (i) to the left of both; (ii) between them; and (iii) to the right of both.

Problem 3

Find the electric field $\vec{E}$ inside a sphere of radius R that carries a charge density proportional to the distance from the origin for some constant r. As a hint you must integrate to get the enclosed charge since the charge density is not uniform.

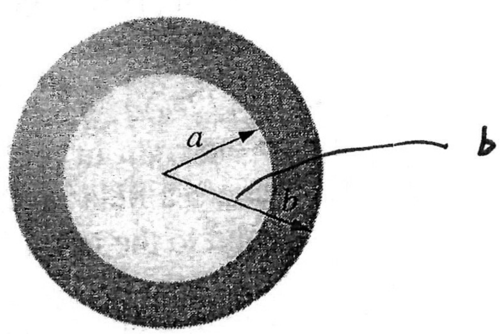
$$\rho = k\,r$$

Problem 4

A thin spherical shell carries a volume charge density ρ given by the following expression

$$\rho = \frac{k}{r^2}$$

in the figure below



where $a \leq r \leq b$ is the figure below. Find the electric field $\vec{E}$ in each of the following three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r < b$. Plot the magnitude of the electric field as a function of r for the case of $b = 2a$.

Problem 5

Consider a long solid cylinder (which could be considered as infinite) which has a radius a. Find the electric field $\vec{E}$ both inside and outside the cylinder. Let the solid have a volume charge density ρ_V which is constant. Note that ρ_V is the volume charge density and ρ is one of our usual coordinates in cylindrical polar coordinates and these two things are not the same!

Problem 6

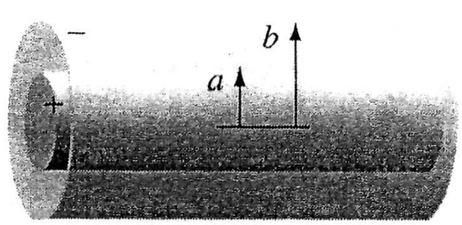
A very long or infinite cylinder carries a volume charge density ρ_V that is proportional to the distance ρ from its axis for some constant k .

$$\rho_V = k\,\rho$$

Note that ρ_V is the volume charge density and ρ is one of our usual coordinates in cylindrical polar coordinates and these two things are not the same! Find the electric field $\vec{E}$ both inside the cylinder. Note that when you perform your integration in cylindrical polar coordinates, please place a prime on the variables of integration to avoid any confusion. You can let the radius of the cylinder be a, but it really not needed in this problem.

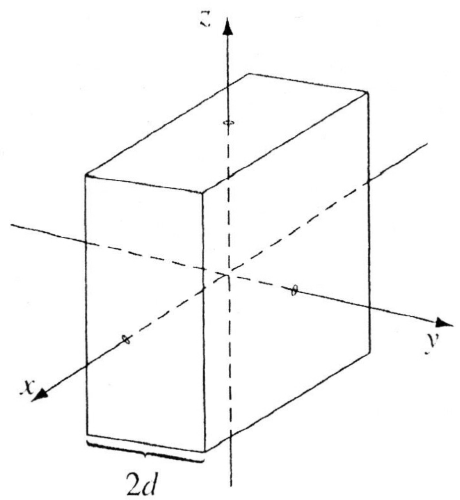
Problem 7

A long coaxial cable is shown in the figure below and it carries a uniform *volume* charge density ρ_V on the inner cylinder of radius a and a uniform *surface* charge density σ on the outer cylindrical shell of radius b. This surface charge is negative and is of just the right magnitude that the cable as a whole is electrically neutral. Find the electric field $\vec{E}$ in each of the following three regions: (i) inside the inner cylinder ($\rho < a$), (ii) between the cylinders ($a < \rho < b$), and (iii) outside the cable ($\rho > b$). Plot the magnitude of the electric field as a function of ρ.



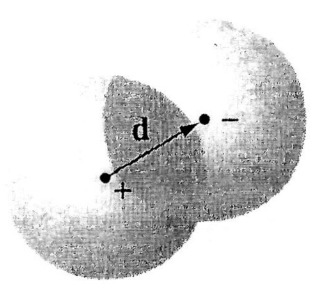
Problem 8

An infinite plane slab of thickness $2d$ carries a uniform volume charge density ρ. Find the electric field $\vec{E}$ as a function of y , where $y = 0$ at the center. Plot the magnitude of the electric field $\vec{E}$ versus y calling E positive when it points in the $+y$ direction and E negative when it points in the $-y$ direction.



Problem 9

Two spheres, each of radius, R, and carrying uniform charge densities ρ and $-\rho$ respectively, are placed so that they partially overlap as seen in the figure below.



Call the vector from the positive center to the negative center $\vec{d}$. Show that the electric field $\vec{E}$ in the region of overlap is constant, and find its value. As a hint use the results of the relevant example discussed in Lecture 7.