USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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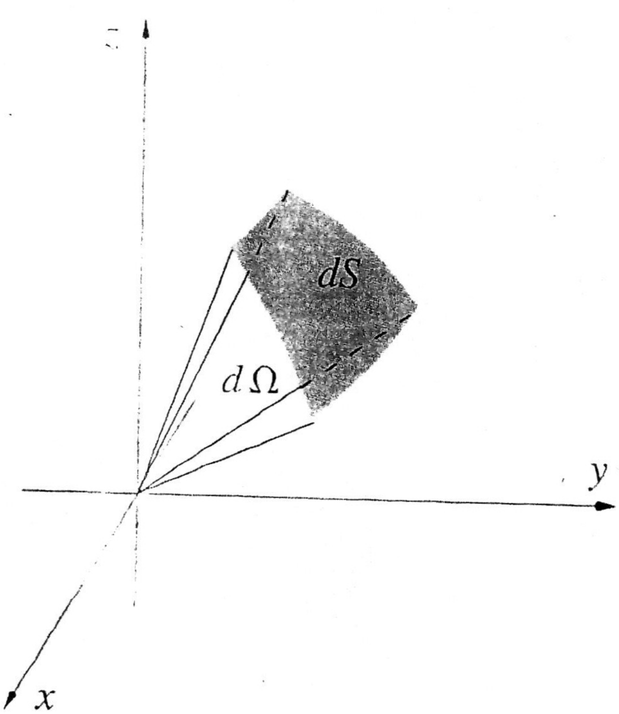
PROBLEM SET VII
(due Tuesday, August 11, 2020)

Problem 1

In Lecture 7 we proved Gauss's Law for a single point charge using a spherical Gaussian surface. We now need to show that Gauss's Law is true for any general closed surface S. Let us first review the idea of a solid angle. Let S be an area on a sphere of radius r centered on the origin. All the rays starting at the origin and passing through S form a cone, which is the solid angle Ω. We say that Ω is subtended by S. The units of solid angles are steradians, just as the units of planar angles are radians. The figure below shows the solid angle $d\Omega$ subtended by dS.



Just as the arc length ds on a circle is related to the angle $d\theta$ in radians that it subtends by $ds = r\,d\theta$ where r is the radius of the circle, dS is related to the solid angle $d\Omega$ (in steradians) that it subtends by $dS = r^2\,d\Omega$. For example, if $r = a =$ constant, then the total surface area of the sphere is $S = 4\pi a^2$ so that a complete solid angle is 4π, just as a complete angle for a circle is 2π.

Now let us apply this to the problem at hand. Consider the figure below for a single point charge centered by Gaussian spherical surface S and a general Gaussian surface S'. All of the appropriate items are defined in the figure below

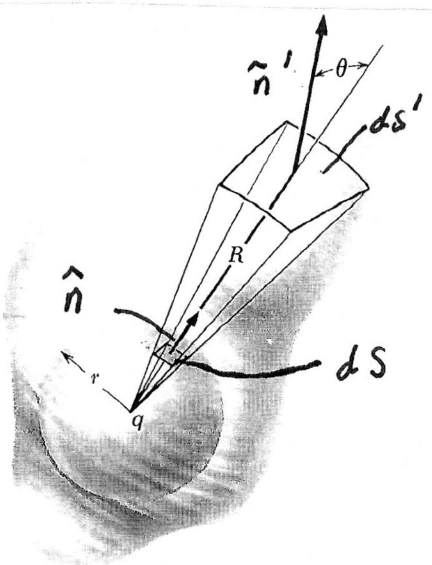


Fig. 1.16 Showing that the flux through *any* closed surface around q is the same as the flux through the sphere.

By carefully applying what we have discussed so far to this figure we realize that

$$\vec{E} = \frac{q\,\hat{r}}{4\pi\epsilon_o\,r^2} \tag{1}$$

$$\vec{E} = \frac{q\,\hat{r}}{4\pi\epsilon_o\,R^2} \tag{2}$$

$$dS = r^2\,d\Omega \tag{3}$$

$$dS' = R^2\, d\Omega \tag{4}$$

Now let us determine the surface integral or flux of $\vec{E}'$ through S'

$$\oint_{S'} \vec{E}' \cdot d\vec{S}' = \oint_{S'} E' dS'\, \hat{r} \cdot \hat{n}' = \oint_{S'} \frac{q}{4\pi\epsilon_o R^2}\, dS'\, \hat{r} \cdot \hat{n}' \tag{5}$$

and since the solid angle Ω subtended by dS' is the same as the solid angle subtended by dS

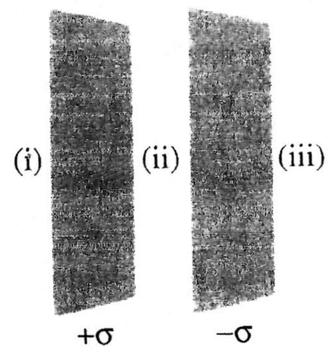
$$\oint_{S'} \vec{E}' \cdot d\vec{S}' = \oint_{S'} \frac{q}{4\pi\epsilon_o r^2} dS\, \hat{r} \cdot \hat{n}' = \oint_{S} \frac{q}{4\pi\epsilon_o r^2}\, dS\, \hat{r} \cdot \hat{n} \tag{6}$$

or where in the last surface integral we realize we are now integrating over S so that $\hat{n}'$ now becomes $\hat{n}$ to obtain the desired result

$$\oint_{S'} \vec{E}' \cdot d\vec{S}' = \oint_{S} \vec{E} \cdot d\vec{S} \tag{7}$$

Problem 2

σ Two infinite parallel planes carry equal but opposite uniform charge densities σ and $-\sigma$ as shown in the figure below.



Find the electric field in each of the three regions: (i) to the left of both; (ii) between them; and (iii) to the right of both.

Problem 3

Find the electric field $\vec{E}$ inside a sphere of radius R that carries a charge density proportional to the distance from the origin for some constant r. As a hint you must integrate to get the enclosed charge since the charge density is not uniform.

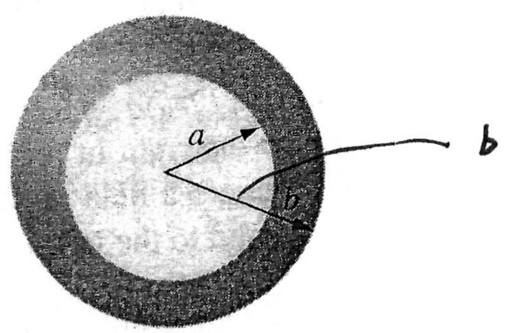
$$\rho = k\,r$$

Problem 4

A thin spherical shell carries a volume charge density ρ given by the following expression

$$\rho = \frac{k}{r^2}$$

in the figure below



where $a \leq r \leq b$ is the figure below. Find the electric field $\vec{E}$ in each of the following three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r < b$. Plot the magnitude of the electric field as a function of r for the case of $b = 2a$.

Problem 5

Consider a long solid cylinder (which could be considered as infinite) which has a radius a. Find the electric field $\vec{E}$ both inside and outside the cylinder. Let the solid have a volume charge density ρ_V which is constant. Note that ρ_V is the volume charge density and ρ is one of our usual coordinates in cylindrical polar coordinates and these two things are not the same!

Problem 6

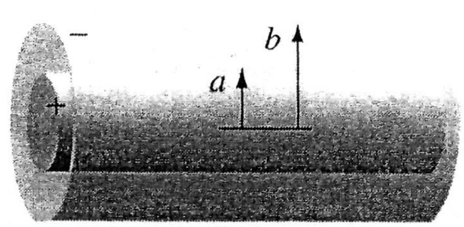
A very long or infinite cylinder carries a volume charge density ρ_V that is proportional to the distance ρ from its axis for some constant k .

$$\rho_V = k\,\rho$$

Note that ρ_V is the volume charge density and ρ is one of our usual coordinates in cylindrical polar coordinates and these two things are not the same! Find the electric field $\vec{E}$ both inside the cylinder. Note that when you perform your integration in cylindrical polar coordinates, please place a prime on the variables of integration to avoid any confusion. You can let the radius of the cylinder be a, but it really not needed in this problem.

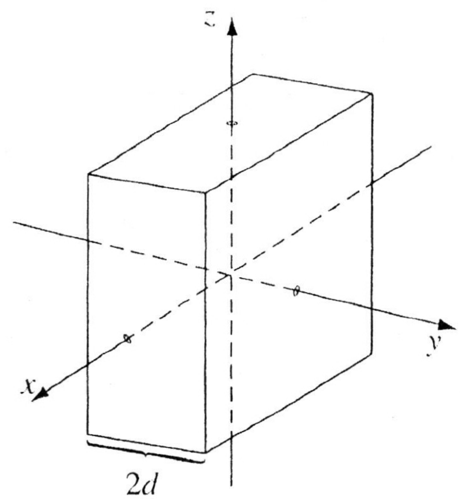
Problem 7

A long coaxial cable is shown in the figure below and it carries a uniform *volume* charge density ρ_V on the inner cylinder of radius a and a uniform *surface* charge density σ on the outer cylindrical shell of radius b. This surface charge is negative and is of just the right magnitude that the cable as a whole is electrically neutral. Find the electric field $\vec{E}$ in each of the following three regions: (i) inside the inner cylinder ($\rho < a$), (ii) between the cylinders ($a < \rho < b$), and (iii) outside the cable ($\rho > b$). Plot the magnitude of the electric field as a function of ρ.



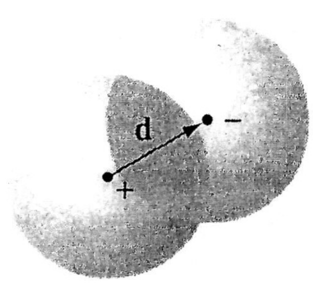
Problem 8

An infinite plane slab of thickness $2d$ carries a uniform volume charge density ρ. Find the electric field $\vec{E}$ as a function of y , where $y = 0$ at the center. Plot the magnitude of the electric field $\vec{E}$ versus y calling E positive when it points in the $+y$ direction and E negative when it points in the $-y$ direction.



Problem 9

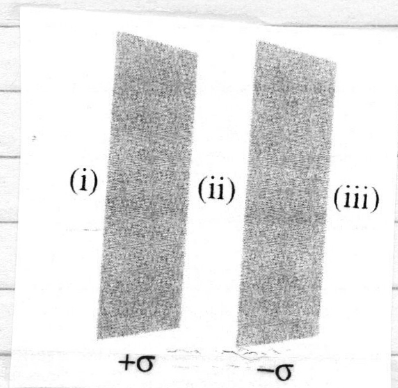
Two spheres, each of radius, R, and carrying uniform charge densities ρ and $-\rho$ respectively, are placed so that they partially overlap as seen in the figure below.



Call the vector from the positive center to the negative center $\vec{d}$. Show that the electric field $\vec{E}$ in the region of overlap is constant, and find its value. As a hint use the results of the relevant example discussed in Lecture 7.

Problem 2

Two infinite parallel planes carry equal but opposite uniform charge densities $\pm \sigma$ as shown in the figure below.



Find the electric field $\vec{E}$ in each of the three regions:

(i) to the left of both

(ii) between them

(iii) to the right of both

Example 2.6. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ (Fig. 2.23). Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.

Solution

The left plate produces a field $(1/2\epsilon_0)\sigma$, which points away from it (Fig. 2.24)—to the left in region (i) and to the right in regions (ii) and (iii). The right plate, being negatively charged, produces a field $(1/2\epsilon_0)\sigma$, which points *toward* it—to the right in regions (i) and (ii) and to the left in region (iii). The two fields cancel in regions (i) and (iii); they conspire in region (ii). *Conclusion:* The field between the plates is σ/ϵ_0, and points to the right; elsewhere it is zero.

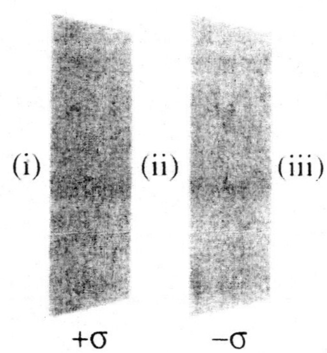


FIGURE 2.23

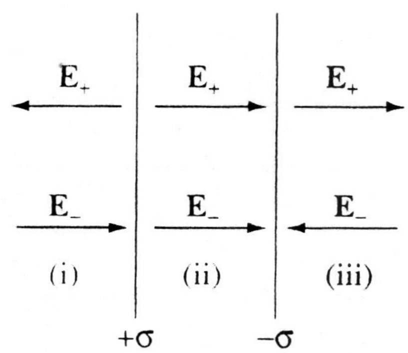


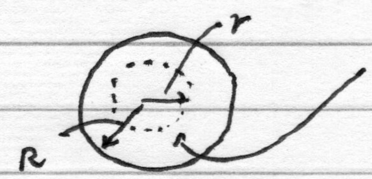
FIGURE 2.24

Problem 3

Find the electric field $\vec{E}$ inside the sphere of radius R that carries a charge density proportional to the distance from the origin

$$\rho = kr$$

for some constant k. You must integrate to get the enclosed charge since the charge density is not uniform. (Hint)



Construct a Gaussian surface (sphere) of radius r ($r < R$) and use Gauss's Law

$$\oint_S \vec{E}\cdot d\vec{S} \xrightarrow[\vec{E} \parallel d\vec{S}]{} \oint_S E\,dS \underset{\substack{E\text{ is}\\ \text{constant}\\ \text{over } S}}{\rightarrow} E\oint_S dS = E\,4\pi r^2$$

$$\frac{q_{en}}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho\, dV = \frac{1}{\epsilon_0}\int_V k r\, r^2 dr \sin\theta\, d\theta\, d\phi$$

$$\frac{q_{en}}{\epsilon_0} = \frac{k}{\epsilon_0} \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^3\,dr\,\sin\theta\,d\theta\,d\phi$$

A little sloppy notation as this is not the r in the integrand

$$\frac{q_{en}}{\epsilon_0} = \frac{4\pi k}{\epsilon_0} \int_0^{r} r^3\,dr = \frac{\pi k}{\epsilon_0} r^4$$

In summary

$$\oint_S \vec{E}\cdot d\vec{s} = E\,4\pi r^2 = \frac{\pi k r^4}{\epsilon_0}$$

or

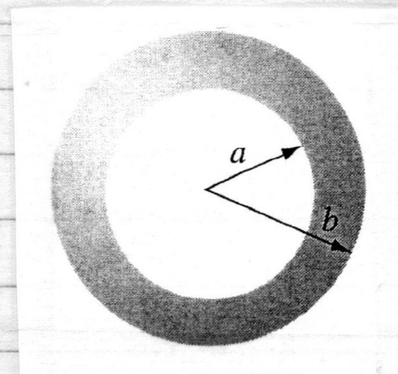
$$\boxed{\vec{E} = \frac{k r^2}{4\epsilon_0}\hat{r}}$$

Problem 4

A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2} \qquad (a \le r \le b)$$

in the figure below



Find the electric field in the three regions:

(i) $r < a$

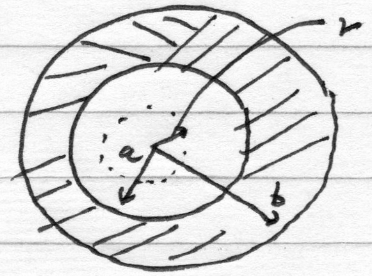
(ii) $a < r < b$

(iii) $r > b$

Plot $|\vec{E}|$ as a function of r for the case of $b = 2a$.

Case I

$r < a$



Construct a Gaussian surface (sphere) of radius r and apply Gauss's Law

$$\oint_S \vec{E}\cdot d\vec{S} \underset{\substack{\vec{E}\parallel d\vec{S} \\ \text{by} \\ \text{symmetry}}}{=} \oint_S E\,dS \xrightarrow[\substack{E \text{ is} \\ \text{constant} \\ \text{over } S}]{} E\oint_S dS \rightarrow E\,4\pi r^2$$

$$\frac{q_{enc}}{\epsilon_0} = 0$$

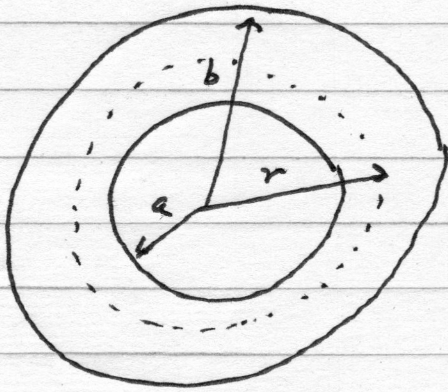
So

$$E\,4\pi r^2 = 0$$

or

$$\vec{E} = 0 \qquad 0 < r < a$$

Case II $\quad a < r < b$



Construct a Gaussian surface (sphere) of radius r and apply Gauss's Law

$$\oint_S \vec{E}\cdot d\vec{s} \xrightarrow[\text{by symmetry}]{\vec{E} \parallel d\vec{s}} \oint_S E\,dS \xrightarrow{E \text{ is constant over } S} E\oint_S dS \longrightarrow E\,4\pi r^2$$

Do not confuse this with r in integral

$$q_{enc} = \int_V \rho\, dV = \int_a^r \frac{k}{r^2}\, r^2 \sin\theta\, d\theta\, d\phi$$

$$q_{enc} = k\,4\pi\,(r-a)$$

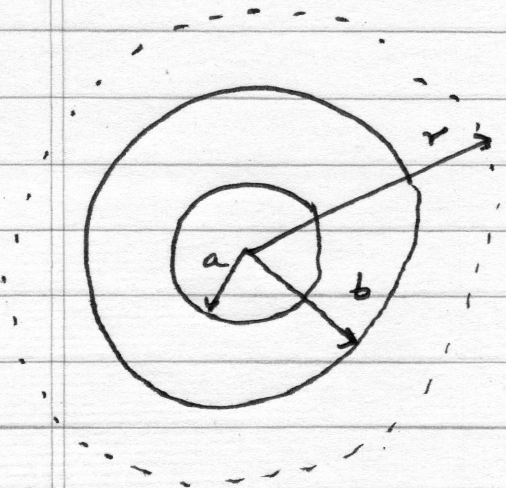
So

$$\oint_S \vec{E}\cdot d\vec{s} = E\,4\pi r^2 = \frac{q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\,4\pi k\,(r-a)$$

or

$$\boxed{\vec{E} = \frac{k}{\epsilon_0}\,\frac{(r-a)\,\hat{r}}{r^2}} \quad a < r < b$$

Case $\underline{III}$ $\qquad r > b$



Construct a Gaussian surface (sphere) of radius r and apply Gauss's Law

$$\oint_S \vec{E}\cdot d\vec{S} \xrightarrow[\text{by symmetry}]{\vec{E} \parallel d\vec{S}} \oint_S E\, dS \xrightarrow{\text{E is constant over S}} E\oint dS$$

$$\Downarrow$$

$$E\, 4\pi r^2$$

$$q_{enc} = \int \rho\, dV = \int_a^b \frac{k}{r^2}\, r^2\, dr \sin\theta\, d\theta\, d\phi$$

(otherwise $\rho = 0$ for $0 < r < a$
for $r > b$

$$q_{enc} = k \int_a^b dr \sin\theta\, d\theta\, d\phi$$

$$q_{enc} = 4\pi k\, (b-a)$$

Therefore

$$E\, 4\pi r^2 = \frac{4\pi k}{\epsilon_0}(b-a) \quad \text{or} \quad \boxed{\vec{E} = \frac{k}{\epsilon_0 r^2}(b-a)\,\hat{r}}$$

Plot $|\vec{E}|$ as a function of r, for the case $b = 2a$.

$$\vec{E} = 0 \quad , \quad 0 < r < a$$

$$\vec{E} = \frac{k}{\epsilon_0} \frac{(r-a)\hat{r}}{r^2} \quad , \quad a < r < b$$

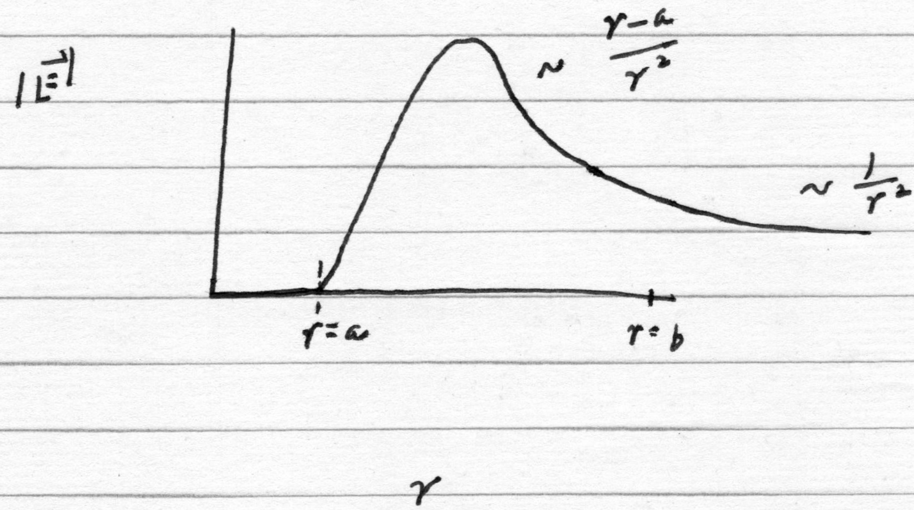
$$\vec{E} = \frac{k\,(b-a)\hat{r}}{\epsilon_0 r^2} \quad , \quad r > b$$

Let $b = 2a$

$$\vec{E} = 0, \quad 0 < r < a$$

$$\vec{E} = \frac{k}{\epsilon_0} \frac{(r-a)}{r^2}\,\hat{r} \quad , \quad a < r < b$$

$$\vec{E} = \frac{k\,a}{\epsilon_0 r^2}\,\hat{r} \quad , \quad r > b$$



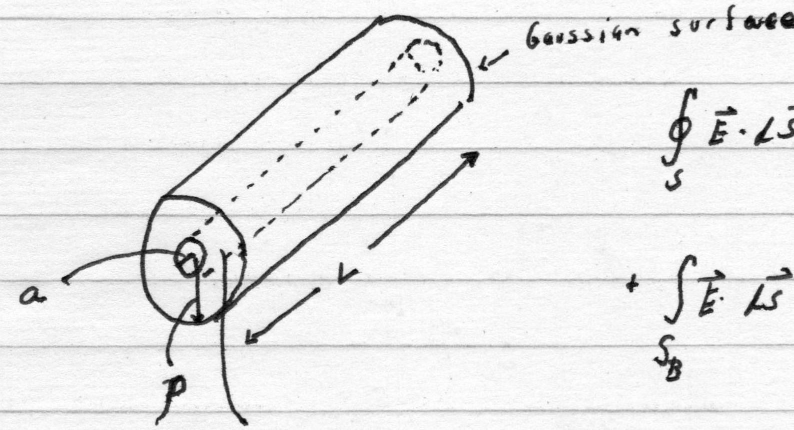
Note that I used Wolfram Alpha to get a better handle on plotting the $a < r < b$ regime. It is a very useful tool, even the free version.

Problem 5

Consider a long cylinder (it could be considered as infinite) which is solid and has radius a. Find the electric field $\vec{E}$ both inside and outside the cylinder. Let the solid have a volume charge density ρ_v which is constant. Note that ρ_v is the volume charge density and ρ is one of our usual coordinates in cylindrical polar coordinates and they are not identical

$\boxed{\rho > a}$

Construct a Gaussian surface (cylinder) of length L and radius a over some region of the solid. This surface has a top, bottom, and side. The flux of $\vec{E}$ through S is given by



$$\oint_S \vec{E}\cdot d\vec{S} = \int_{S_T} \vec{E}\cdot d\vec{S}$$

$$+ \int_{S_B} \vec{E}\cdot d\vec{S} + \int_{S} \vec{E}\cdot d\vec{S}$$

By symmetry $\vec{E}$ is radially outward from the cylinder so

$$\vec{E} \perp \vec{S}_T \quad \text{and} \quad \vec{E} \perp \vec{S}_B$$

and any contribution to the flux must vanish.

$$\oint_S \vec{E} \cdot d\vec{S} = \int_S \vec{E} \cdot d\vec{S}$$

Along the side of the Gaussian cylinder $\vec{E} \parallel d\vec{S}$ and it is clear by symmetry that $\vec{E}$ is a constant over S so

$$\oint_S \vec{E} \cdot d\vec{S} = \int_S E\,dS = E\int_S dS = E\ 2\pi\rho L$$

The enclosed charge is

$$\frac{q_{en}}{\epsilon_0} = \int_V \frac{\rho_v}{\epsilon_0}\,dV = \frac{\rho_v}{\epsilon_0}\int_V dV$$

and in cylindrical polar coordinates

$$\frac{q_{en}}{\epsilon_0} = \frac{\rho_v}{\epsilon_0}\int_0^a \rho\,d\rho\ d\phi\ dz$$

$$\frac{q_{en}}{\epsilon_0} = \frac{\rho_v a^2}{2\epsilon_0} L\ 2\pi$$

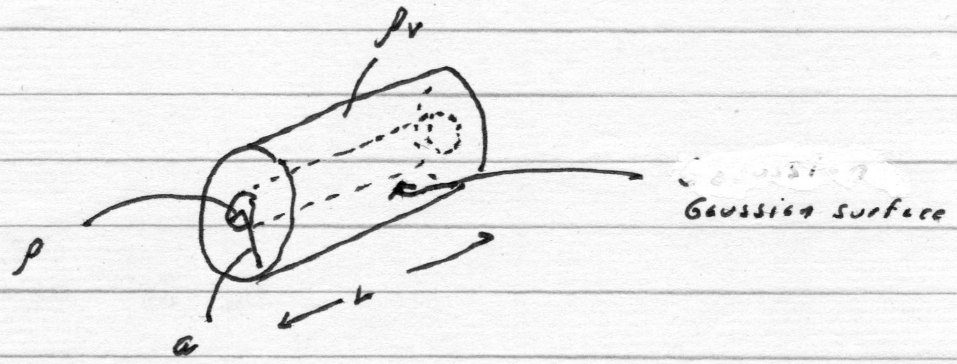
Thus

$$E\ 2\pi\rho L = \frac{\rho_v a^2}{2\epsilon_0} L\ 2\pi$$

or

$$\boxed{\vec{E} = \frac{\rho_v}{2\epsilon_0}\frac{a^2}{\rho}\hat{\rho}}$$

For $\rho < a$ most of everything we have done still holds. We construct a Gaussian cylinder of radius ρ inside the solid so $\rho < a$



For example

$$\oint_S \vec{E} \cdot d\vec{S} = E 2\pi L \rho$$

but the enclosed charge is different

$$\frac{q_{en}}{\epsilon_0} = \frac{\rho_v}{\epsilon_0} \int_V dV = \frac{\rho_v}{\epsilon_0} \int_0^L \int_0^{2\pi} \int_0^{\rho} \rho' d\rho' d\phi' dz'$$

where note we have introduced primes for the variables of integration so that ρ and ρ' are not confused

$$\frac{q_{en}}{\epsilon_0} = \frac{\rho_v \rho^2}{2\epsilon_0} 2\pi L \quad \text{so}$$

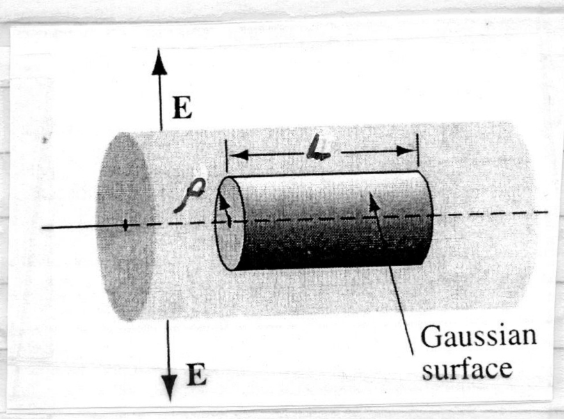
$$\boxed{\vec{E} = \frac{\rho_v \rho \, \hat{\rho}}{2\epsilon_0}}$$

Problem 6

A very long or infinite cylinder carries a charge density that is proportional to the distance from the axis

$$\rho_v = k\rho$$

for some constant k. Note that in this problem ρ_v is the volume charge density and ρ is the distance or one of the coordinates in cylindrical polar coordinates. Find $\vec{E}$ inside the cylinder. When you perform your integration in cylindrical polar coordinates please place a prime on the variables of integration to avoid any confusion. You can let the radius of the cylinder be a but you will not need it in this problem.



We can utilize much of the machinery from the previous problem

$$\oint_S \vec{E} \cdot d\vec{S} = \oint_S \vec{E} \cdot d\vec{S} = \int_{S_{SIDE}} \vec{E} \cdot d\vec{S} = E\,2\pi L r$$

$$\frac{q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0} \int \rho_v \, dV = \frac{1}{\epsilon_0} \int_0^{\rho} k\rho' \rho' \, d\rho' \, d\phi' \, dz'$$

$$\frac{q_{enc}}{\epsilon_0} = \frac{k}{\epsilon_0} \left[\frac{(\rho')^3}{3} L \, 2\pi \right]_0^{\rho}$$

$$\frac{q_{enc}}{\epsilon_0} = \frac{2\pi L k}{3\epsilon_0} \rho^3$$

Thus

$$E\,2\pi L r = \frac{2\pi L k}{3\epsilon_0} \rho^3$$

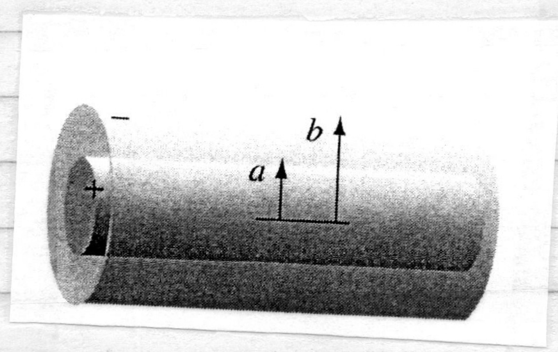
$$E = \frac{k\rho^2}{3\epsilon_0}$$

$$\vec{E} = \frac{k\rho^2}{3\epsilon_0} \hat{\rho} \quad , \quad \rho \text{ inside cylinder}$$

Again we use primes for variables of integration so $\rho' \neq \rho$!

Problem 7

A long coaxial cable is shown in the figure below and it carries a uniform volume charge density ρ_v on the inner cylinder of radius a and a and a uniform surface charge density σ on the outer shell of radius b. This surface charge is negative and is of just the right magnitude that the cable as a whole is electrically neutral.



Find the electric field $\vec{E}$ in each of the following three regions:

(i) inside the inner cylinder $(\rho < a)$

We can use all of the machinery from the previous problems using Gauss's Law and cylindrical symmetry.

For $\rho < a$

$$\oint_S \vec{E} \cdot d\vec{s} = E 2\pi L \rho = \frac{q_{en}}{\epsilon_0} = \frac{\pi L \rho_v^2}{\epsilon_0}$$

$$\boxed{\vec{E} = \frac{1}{2\epsilon_0} \rho \rho_v \hat{\rho}}$$

For $a < \rho < b$

$$\oint_S \vec{E} \cdot d\vec{s} = E 2\pi L \rho = \frac{q_{en}}{\epsilon_0} = \rho_v \frac{a^2 2\pi L}{2\epsilon_0}$$

$$\boxed{\vec{E} = \frac{\rho_v a^2}{2\epsilon_0 \rho} \hat{\rho}}$$

Note that there is no charge for $a < \rho < b$

For $\rho > b$ [outside the cable]

The first part is relatively easy using our previous knowledge and skills. The Gaussian surface (cylinder) has a radius $\rho > b$

$$\oint_S \vec{E} \cdot d\vec{S} = E 2\pi L \rho$$

We have to be careful with the charge enclosed

$$\frac{q_{en}}{\epsilon_0} = \frac{1}{\epsilon_0}\left[\iint_V \rho_v \, dV + \int_S \sigma \, dS\right]$$

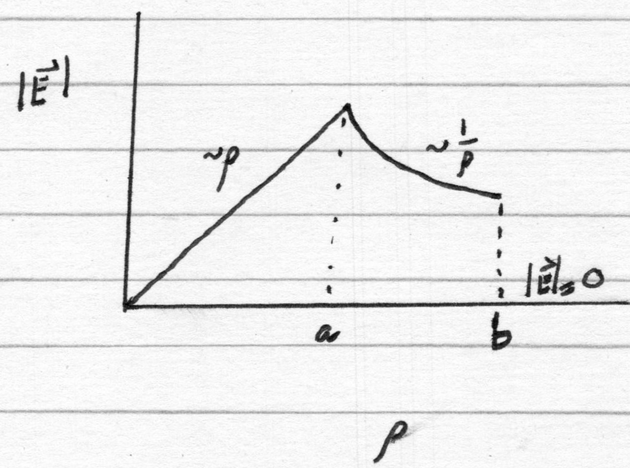
where

$$\rho_v = \frac{q}{\text{Volume of enclosed cylinder}} \qquad \sigma = \frac{q'}{\text{Surface of enclosed cylinder}}$$

however $q = -q'$ for electrical neutrality!

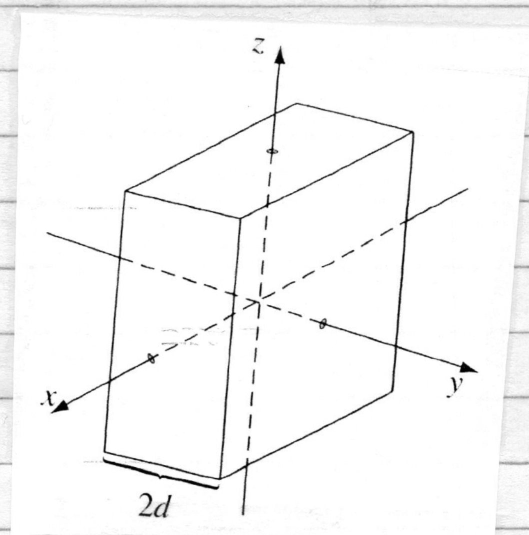
Thus $$\boxed{\vec{E} = 0 \quad (\rho > b)}$$

Plot $|\vec{E}|$ as a function of ρ

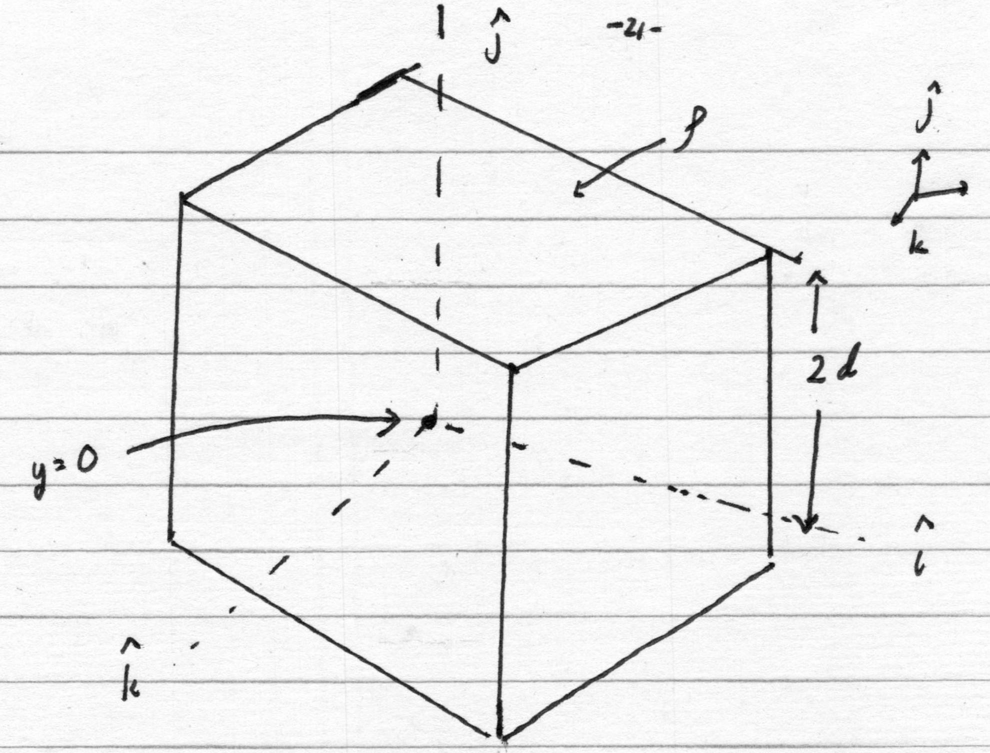


Problem 8

An infinite plane slab of thickness $2d$ carries a uniform volume charge density ρ as illustrated in the figure below

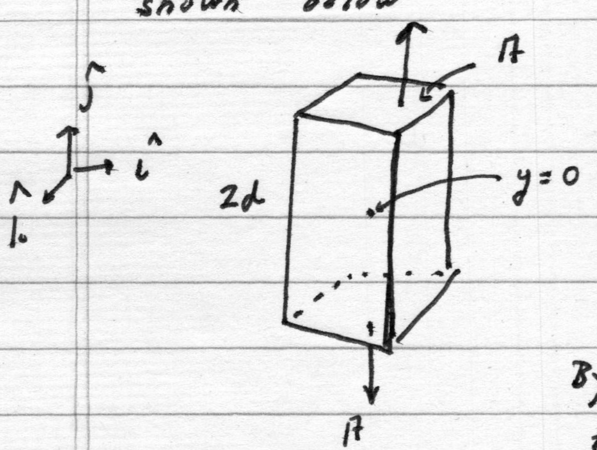


Find the electric field as a function of y, where $y = 0$ at the center. Plot E versus y, calling E positive when it points in the $+y$ direction and negative when it points in the $-y$ direction.



In the above illustration we show a finite piece of the slab as discerned from our original figure where the center is at $y=0$. Note that $-d < y < d$ for the slab.

We will need a Gaussian surface (pill box) to find $\vec{E}$. Such a suitable candidate is shown below



Clearly for this choice

Where
$y \geq d$
$y \leq -d$
and our box encompasses a region completely full of volume charge density ρ.

By symmetry the $\vec{E}$ field is in $\pm\hat{j}$ direction so

$$\oint_S \vec{E} \cdot d\vec{S} = 2EA = \frac{q_{en}}{\epsilon_0} = \rho \frac{2dA}{\epsilon_0}$$

Thus for -

$$\vec{E} = +\hat{j}\,\frac{\rho d}{\epsilon_0} \qquad y > d$$

$$\vec{E} = -\hat{j}\,\frac{\rho d}{\epsilon_0} \qquad y < -d$$

which are both constants.

Now for a smaller Gaussian pill box of height less than $2d$ we are exploring the regime

$$-d < y < d$$

We now have

$$\oint_S \vec{E}\cdot d\vec{S} = 2EA = \frac{q_{en}}{\epsilon_0} = \frac{\rho}{\epsilon_0}\,2yA$$

or

$$E = \frac{\rho}{\epsilon_0}\,y$$

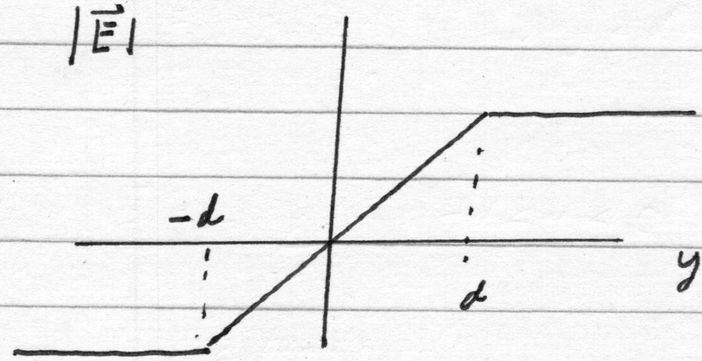
or

$$\vec{E} = +\frac{\rho}{\epsilon_0}\,y\,\hat{j} \quad , \quad 0 < y < d$$

$$\vec{E} = -\frac{\rho}{\epsilon_0}\,y\,\hat{j} \quad , \quad -d < y < 0$$

where the pillbox goes in height from y to $-y$ or it has height of $2y$

We can plot all of this along the y-axis

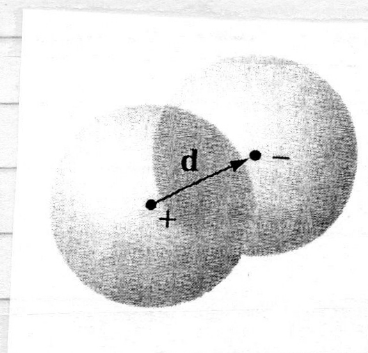


Problem 9

Two spheres, each of radius R and carrying uniform volume charge densities

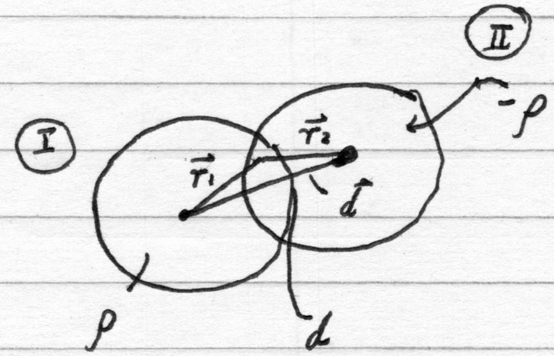
$$+\rho \text{ and } -\rho$$

respectively, are placed so that they partially overlap as seen in the figure below.

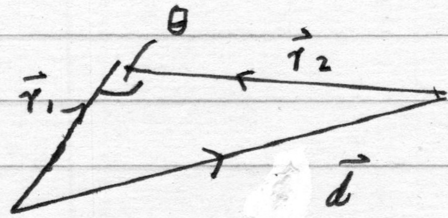


Call the vector from the positive center to the negative center $\vec{d}$. Show that the electric field $\vec{E}$ in the region of overlap is constant, and find its value.

[Hint: Use the answer to Problem discussed in Lecture 7]



Let us blow up our triangle here



and note that according to the Law of Cosines

$$d^2 = r_1^{\,2} + r_2^{\,2} - 2r_1 r_2 \cos\theta$$

Now the electric field anywhere inside the region of overlap due to ρ is according to Problem

$$\vec{E}_{I} = \frac{\vec{r}_1\,\rho}{3\epsilon_0}$$

and for $-\rho$

$$\vec{E}_{II} = \frac{-\vec{r}_2\,\rho}{3\epsilon_0}$$

Using the principle of superposition the total electric field inside the region of overlap due to these two sources is

$$\vec{E}_T = \vec{E}_I + \vec{E}_{II}$$

$$\vec{E}_T = \frac{\vec{r}_1 \rho - \vec{r}_2 \rho}{3\epsilon_0}$$

and

$$|E_T| = \left[\frac{r_1^2 \rho^2 + r_2^2 \rho^2 - 2 r_1 r_2 \rho^2 \cos\theta}{[3\epsilon_0]^2}\right]^{1/2}$$

but from the Law of Cosines applied to our triangle this is

$$\boxed{|E_T| = \left[\frac{\rho^2 d^2}{[3\epsilon_0]^2}\right]^{1/2} = \frac{\rho d}{3\epsilon_0}}$$

which is a constant!