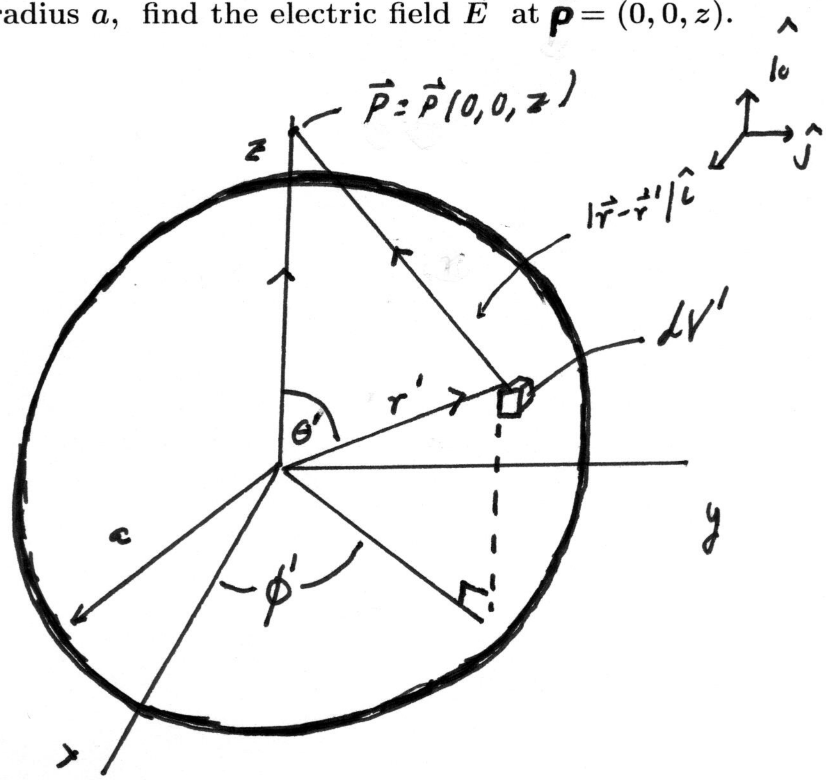
USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
Dr. Steven L. Richardson (srichards22@comcast.net)
Professor Emeritus of Electrical Engineering, Department of Electrical and Computer Engineering, Howard University, Washington, DC
and
Faculty Associate in Applied Physics, John A. Paulson School of Engineering and Applied Sciences, Harvard University, Cambridge, MA

PROBLEM SET VI
(due Tuesday, August 4, 2020)

Problem 1

Given a spherical volume charge distribution with uniform charge density ρ and radius a, find the electric field $\vec{E}$ at $\vec{p} = (0, 0, z)$.



Clearly we will work in spherical polar coordinates here where the primed variables refer to the source of charge and we will introduce the variable s to simplify our calculation

$$\vec{E} = \int \frac{dq' \, \hat{r}}{4\pi\epsilon_o |\vec{r} - \vec{r'}|^2} = \int \frac{dq' \, \hat{r}}{4\pi\epsilon_o s^2} \tag{1}$$

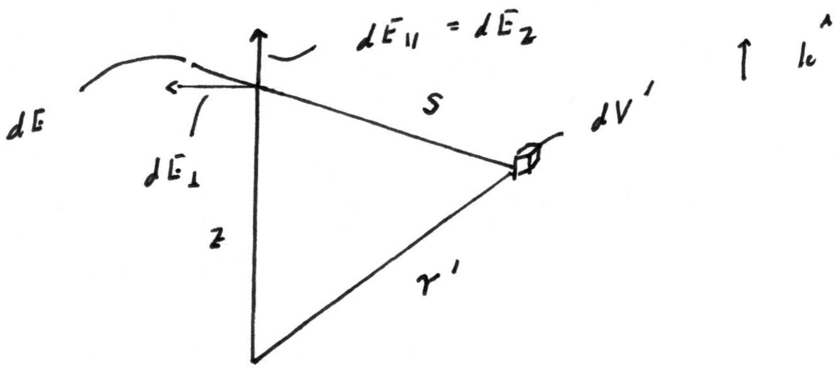
Let us make things easy for ourselves and use symmetry here

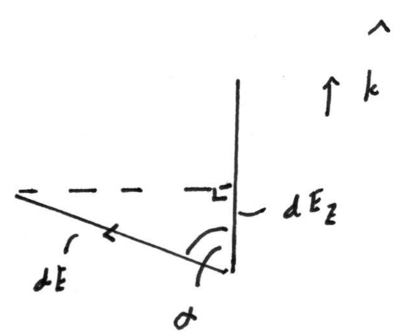
$$\vec{E} = \int d\vec{E} \tag{2}$$

When we look at $\vec{P}$ clearly $d\vec{E}$ or dE can be broken down into its two components

$$dE = dE_{\perp} + dE_z \tag{3}$$

All of the $dE_{\perp}$ components add up to zero by symmetry once you integrate over the entire sphere so only the dE_z component remains where





and

$$dE_z = dE \, \cos\alpha \tag{4}$$

Thus we need to find

$$E_z = \int dE_z = \int \cos\alpha \, dE \tag{5}$$

or

$$E_z = \int dE \, \cos\alpha = \int \frac{dq' \cos\alpha}{4\pi\epsilon_o s^2} \tag{6}$$

or

$$E_z = \int dE \, \cos\alpha = \rho \int \frac{dV' \cos\alpha}{4\pi\epsilon_o s^2} \tag{7}$$

In spherical polar coordinates

$$dV' = (r')^2 \, dr' \, \sin\theta' \, d\theta' \, d\phi' \tag{8}$$

and our desired electric field component E_z becomes

$$E_z = \rho \int_V \frac{(r')^2 \, dr' \, \sin\theta' \, d\theta' \, d\phi' \cos\alpha}{4\,\pi\,\epsilon_o\,s^2} \tag{9}$$

where we have set $\rho = \rho'$. Note that this ρ is a volume charge density and not a distance as used in cylindrical polar coordinates!

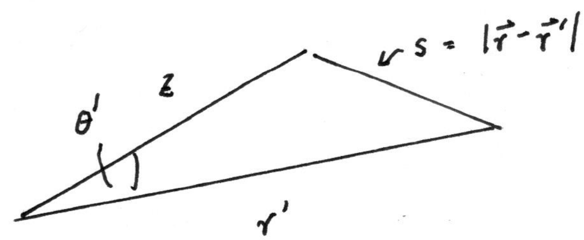
Now this integral depends on α, θ', ϕ', s, and r' . We shall write it in terms of θ', ϕ', s, and r' only. Note also that z is fixed in this problem. The Law of Cosines can help us out here using the figure below

$$s^2 = z^2 + (r')^2 - 2\,z\,r' \, \cos\theta' \tag{10}$$

or

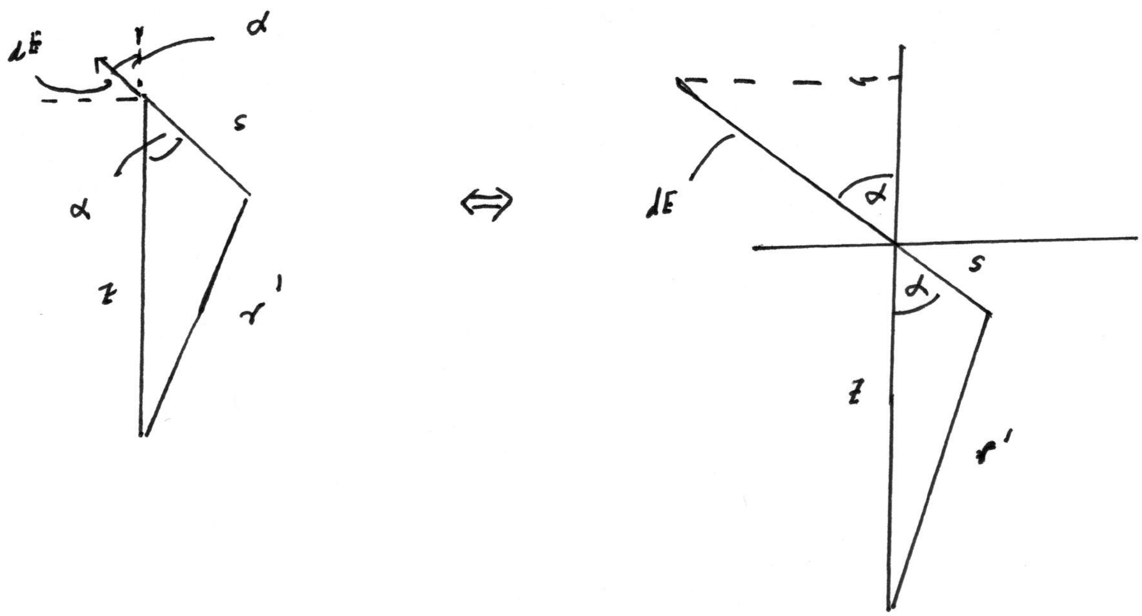
$$2\,z\,r' \, \cos\theta' = z^2 + (r')^2 - s^2 \tag{11}$$

or



$$\cos\theta' = \frac{[z^2 + (r')^2 - s^2]}{2\,z\,r'} \tag{12}$$

Next let us express $\alpha = \alpha(s, r')$ again using the Law of Cosines



$$(r')^2 = s^2 + z^2 - 2\,z\,s\,\cos\alpha \tag{13}$$

or

$$\cos\alpha = \frac{z^2 + s^2 - (r')^2}{2\,z\,s} \tag{14}$$

Now by placing Eq. (1-14) into Eq. (1-9) we have eliminated α to yield

$$E_z = \frac{\rho}{4\pi\epsilon_o}\int_V \frac{(r')^2\,dr'\,\sin\theta'\,d\theta'\,d\phi'}{s^2}\;\frac{[z^2 + s^2 - (r')^2]}{2\,z\,s} \tag{15}$$

Next it is easier to do the integral over the azimuthal angle ϕ' first

$$E_z = \frac{\rho}{2\epsilon_o} \int\int \frac{(r')^2\, dr' \, \sin\theta' \, d\theta}{s^2} \; \frac{[z^2 + s^2 - (r')^2]}{2\, z\, s} \tag{16}$$

Now let us tackle the term $\sin\theta' \, d\theta'$. When we integrate over the polar angle θ' we realize from our original figure that r' is fixed but s is not, as it is a variable. Note also that we treat z as a constant when we do the integral over θ' since we are evaluating the electric field $\vec{E}(z)$ at a fixed value of z . Let us apply these three facts to Eq. (1-10)

$$\cos\theta' = \frac{[z^2 + (r')^2 - s^2]}{2\, z\, r'} \tag{17}$$

$$-\sin\theta' d\theta' = \frac{-s\, ds}{2\, z\, r'} \tag{18}$$

$$\sin\theta' d\theta' = \frac{s\, ds}{z\, r'} \tag{19}$$

Show that Eq. (1-16) becomes

$$E_z = \frac{\rho}{4\epsilon_o\, z^2} \int\int \frac{(r')^2\, dr'}{s^2\, r'} \; \left[z^2 + s^2 - (r')^2\right] \, ds \tag{20}$$

or

$$E_z = \frac{\rho}{4\epsilon_o\, z^2} \int\int r' \left[1 + \frac{z^2 - (r')^2}{s^2}\right] \, ds\, dr' \tag{21}$$

Now integrate the inner integral over s and use Eq. (1-12) to find the appropriate limits of integration as the polar angle goes from $\theta = 0$ to $\theta' = \pi$.

You should get a result of $4(r')^2$ for this integral. Finally do the integral over r' from $r' = 0$ to $r' = a$. You should obtain the final answer

$$\vec{E}(z) = \int \frac{q}{4\pi\epsilon_o z^2}\,\hat{k} \;=\; \vec{E}(r) = \int \frac{q}{4\pi\epsilon_o r^2}\,\hat{r} \tag{22}$$

where q is the total charge of the sphere. Note that your answer is identical to the case where the total charge q is concentrated at the center of the sphere which is neat! Considering how difficult this problem is to do you will appreciate later in Lecture 7 how Gauss's Law can be used to simplify your calculation.

Problem 2

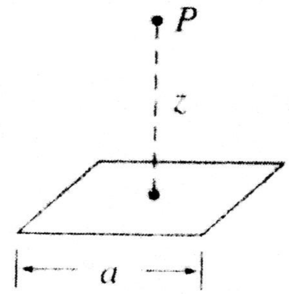
In Lecture 6 we calculated the electric field $\vec{E}$ at a point $\vec{P}$ above the end of a half-infinite line of linear charge density λ . We discovered the remarkable observation that $\vec{E}$ is always pointed up at an angle of 45° independent of the value of z. Repeat this calculation and then look at the same problem for the case of the other possible half-infinite line of linear charge density λ . Use these two separate results and the principle of superposition to get the expected result for the infinite line of linear charge density λ as discussed in Lecture 5.

Problem 3

In Lecture 6 we calculated the electric field $\vec{E}$ of a thin plastic rod bent into a semicircle of radius a with a linear charge density $\lambda = \frac{q}{2\pi a}$. We found $\vec{E}$ at the center of the circle. Repeat this calculation and then look at the same problem for the case of another thin plastic rod bent into the other semicircle of radius a with a linear charge density $\lambda = \frac{q}{2\pi a}$. Use these two separate results and the principle of superposition to get the expected result for the circular thin rod of linear charge density λ as discussed in Lecture 5.

Problem 4

Show that the electric field $\vec{E}$ a distance z above the center of a square loop of side a carrying a uniform linear charge density λ is



$$\vec{E}(z) = \frac{\lambda\, a\, z}{\pi\, \epsilon_o\, \left(z^2 + \frac{a^2}{4}\right) \sqrt{z^2 + \frac{a^2}{2}}}\, \hat{k}$$

Hint: Use the results of Problem 6 in Problem Set V and the following concepts: the principle of superposition, vector analysis, and trigonometry. This problem is simply one where geometry is what you have to pay attention to!

Problem 5

Show that the curl of the gradient of a scalar field vanishes.

Problem 6

Show that divergence of the curl of a vector field vanishes.

Problem 7

See if you can express the divergence of the gradient in a fairly simple form.