USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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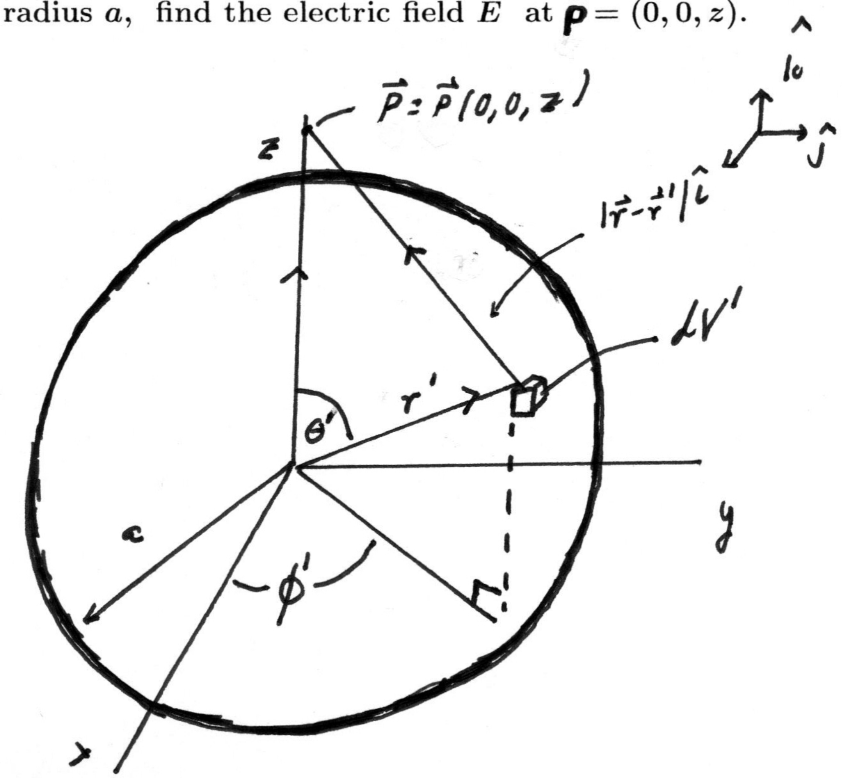
PROBLEM SET VI
(due Tuesday, August 4, 2020)

Problem 1

Given a spherical volume charge distribution with uniform charge density ρ and radius a, find the electric field $\vec{E}$ at $\vec{p} = (0, 0, z)$.



Clearly we will work in spherical polar coordinates here where the primed variables refer to the source of charge and we will introduce the variable s to simplify our calculation

$$\vec{E} = \int \frac{dq'\,\hat{r}}{4\pi\epsilon_o|\vec{r} - \vec{r'}|^2} = \int \frac{dq'\,\hat{r}}{4\pi\epsilon_o s^2} \tag{1}$$

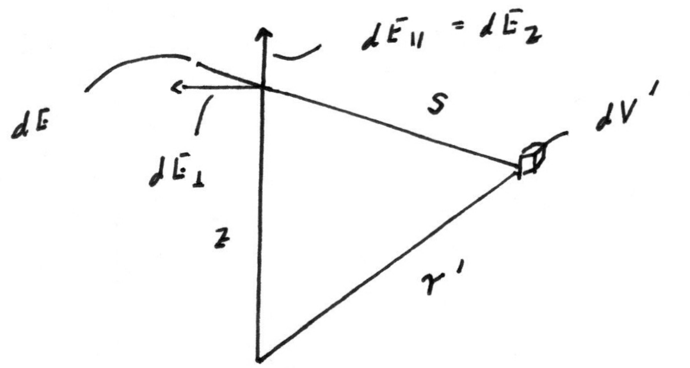
Let us make things easy for ourselves and use symmetry here

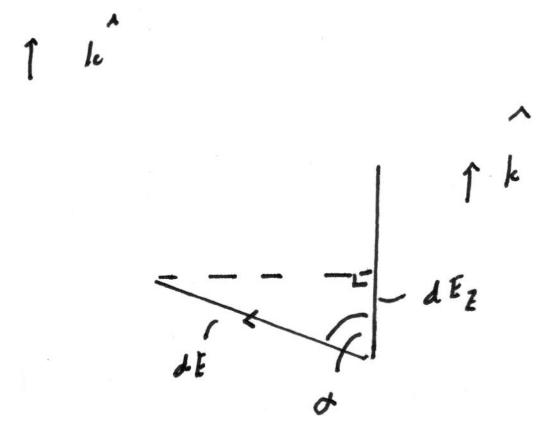
$$\vec{E} = \int d\vec{E} \tag{2}$$

When we look at $\vec{P}$ clearly $d\vec{E}$ or dE can be broken down into its two components

$$dE = dE_\perp + dE_z \tag{3}$$

All of the $dE_\perp$ components add up to zero by symmetry once you integrate over the entire sphere so only the dE_z component remains where





and

$$dE_z = dE\,\cos\alpha \tag{4}$$

Thus we need to find

$$E_z = \int dE_z = \int \cos\alpha\, dE \tag{5}$$

or

$$E_z = \int dE \, \cos\alpha = \int \frac{dq' \cos\alpha}{4\pi\epsilon_o s^2} \tag{6}$$

or

$$E_z = \int dE \, \cos\alpha = \rho \int \frac{dV' \cos\alpha}{4\pi\epsilon_o s^2} \tag{7}$$

In spherical polar coordinates

$$dV' = (r')^2 \, dr' \, \sin\theta' \, d\theta' \, d\phi' \tag{8}$$

and our desired electric field component E_z becomes

$$E_z = \rho \int_V \frac{(r')^2 \, dr' \, \sin\theta' \, d\theta' \, d\phi' \cos\alpha}{4\,\pi\,\epsilon_o\,s^2} \tag{9}$$

where we have set $\rho = \rho'$. Note that this ρ is a volume charge density and not a distance as used in cylindrical polar coordinates!

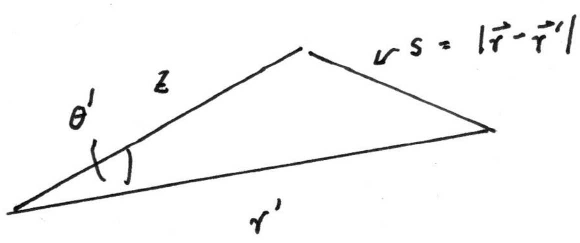
Now this integral depends on α, θ', ϕ', s, and r' . We shall write it in terms of θ', ϕ', s, and r' only. Note also that z is fixed in this problem. The Law of Cosines can help us out here using the figure below

$$s^2 = z^2 + (r')^2 - 2\,z\,r' \cos\theta' \tag{10}$$

or

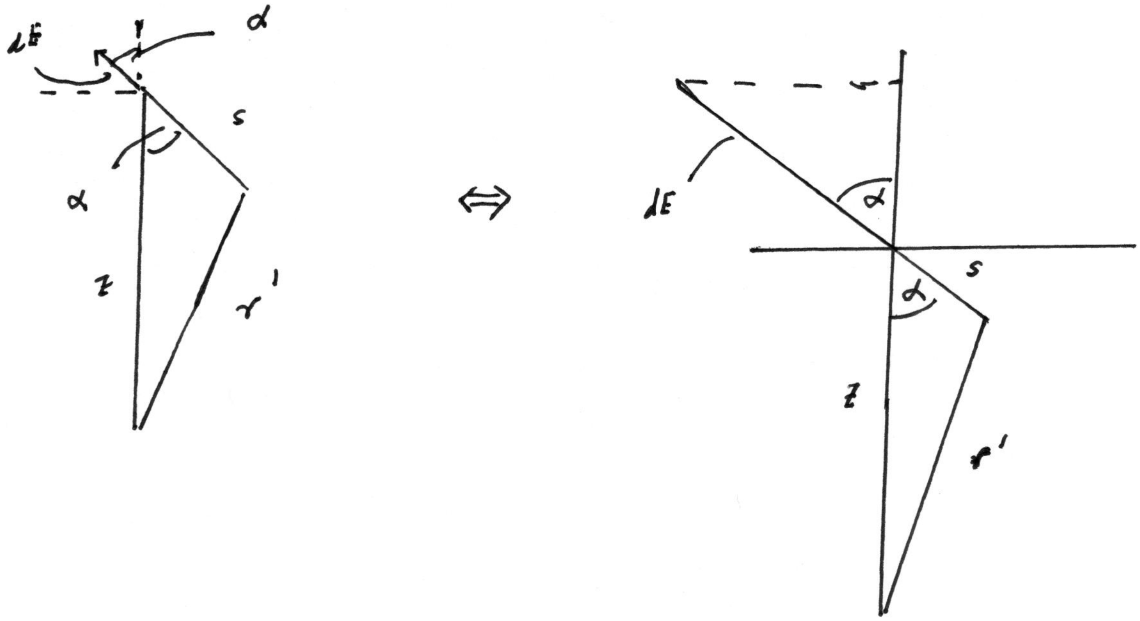
$$2\,z\,r' \cos\theta' = z^2 + (r')^2 - s^2 \tag{11}$$

or



$$\cos\theta' = \frac{[z^2 + (r')^2 - s^2]}{2\,z\,r'} \tag{12}$$

Next let us express $\alpha = \alpha(s, r')$ again using the Law of Cosines



$$(r')^2 = s^2 + z^2 - 2\,z\,s\,\cos\alpha \tag{13}$$

or

$$\cos\alpha = \frac{z^2 + s^2 - (r')^2}{2\,z\,s} \tag{14}$$

Now by placing Eq. (1-14) into Eq. (1-9) we have eliminated α to yield

$$E_z = \frac{\rho}{4\pi\epsilon_o} \int_V \frac{(r')^2\,dr'\,\sin\theta'\,d\theta'\,d\phi'}{s^2}\;\frac{[z^2 + s^2 - (r')^2]}{2\,z\,s} \tag{15}$$

Next it is easier to do the integral over the azimuthal angle ϕ' first

$$E_z = \frac{\rho}{2\epsilon_o} \int\int \frac{(r')^2\, dr' \,\sin\theta' \,d\theta}{s^2} \,\frac{[z^2 + s^2 - (r')^2]}{2\,z\,s} \tag{16}$$

Now let us tackle the term $\sin\theta'\, d\theta'$. When we integrate over the polar angle θ' we realize from our original figure that r' is fixed but s is not, as it is a variable. Note also that we treat z as a constant when we do the integral over θ' since we are evaluating the electric field $\vec{E}(z)$ at a fixed value of z . Let us apply these three facts to Eq. (1-10)

$$\cos\theta' = \frac{[z^2 + (r')^2 - s^2]}{2\,z\,r'} \tag{17}$$

$$-\sin\theta' d\theta' = \frac{-s\, ds}{2\,z\,r'} \tag{18}$$

$$\sin\theta' d\theta' = \frac{s\, ds}{z\,r'} \tag{19}$$

Show that Eq. (1-16) becomes

$$E_z = \frac{\rho}{4\epsilon_o\, z^2} \int\int \frac{(r')^2\, dr'}{s^2\, r'} \left[z^2 + s^2 - (r')^2\right] ds \tag{20}$$

or

$$E_z = \frac{\rho}{4\epsilon_o\, z^2} \int\int r' \left[1 + \frac{z^2 - (r')^2}{s^2}\right] ds\, dr' \tag{21}$$

Now integrate the inner integral over s and use Eq. (1-12) to find the appropriate limits of integration as the polar angle goes from $\theta = 0$ to $\theta' = \pi$.

You should get a result of $4(r')^2$ for this integral. Finally do the integral over r' from $r' = 0$ to $r' = a$. You should obtain the final answer

$$\vec{E}(z) = \int \frac{q}{4\pi\epsilon_o z^2}\,\hat{k} \;=\; \vec{E}(r) = \int \frac{q}{4\pi\epsilon_o r^2}\,\hat{r} \tag{22}$$

where q is the total charge of the sphere. Note that your answer is identical to the case where the total charge q is concentrated at the center of the sphere which is neat! Considering how difficult this problem is to do you will appreciate later in Lecture 7 how Gauss's Law can be used to simplify your calculation.

Problem 2

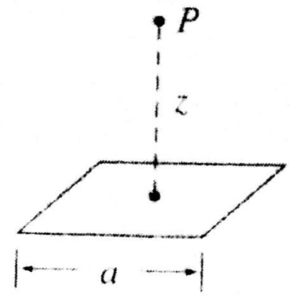
In Lecture 6 we calculated the electric field $\vec{E}$ at a point $\vec{P}$ above the end of a half-infinite line of linear charge density λ . We discovered the remarkable observation that $\vec{E}$ is always pointed up at an angle of 45° independent of the value of z. Repeat this calculation and then look at the same problem for the case of the other possible half-infinite line of linear charge density λ . Use these two separate results and the principle of superposition to get the expected result for the infinite line of linear charge density λ as discussed in Lecture 5.

Problem 3

In Lecture 6 we calculated the electric field $\vec{E}$ of a thin plastic rod bent into a semicircle of radius a with a linear charge density $\lambda = \frac{q}{2\pi a}$. We found $\vec{E}$ at the center of the circle. Repeat this calculation and then look at the same problem for the case of another thin plastic rod bent into the other semicircle of radius a with a linear charge density $\lambda = \frac{q}{2\pi a}$. Use these two separate results and the principle of superposition to get the expected result for the circular thin rod of linear charge density λ as discussed in Lecture 5.

Problem 4

Show that the electric field $\vec{E}$ a distance z above the center of a square loop of side a carrying a uniform linear charge density λ is



$$\vec{E}(z) = \frac{\lambda\, a\, z}{\pi\, \epsilon_o \left(z^2 + \frac{a^2}{4}\right)\sqrt{z^2 + \frac{a^2}{2}}}\ \hat{k}$$

Hint: Use the results of Problem 6 in Problem Set V and the following concepts: the principle of superposition, vector analysis, and trigonometry. This problem is simply one where geometry is what you have to pay attention to!

Problem 5

Show that the curl of the gradient of a scalar field vanishes.

Problem 6

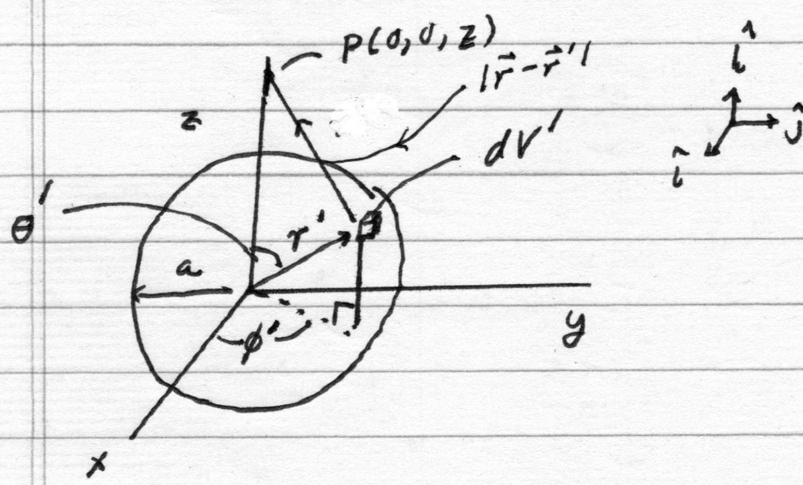
Show that divergence of the curl of a vector field vanishes.

Problem 7

See if you can express the divergence of the gradient in a fairly simple form.

Problem 1

Given a spherical volume charge distribution with uniform charge density ρ and radius a, find the electric field $\vec{E}$ at $P(0, 0, z)$.



Clearly we will work in spherical polar coordinates here where the primed variables refer to the source of charge and $|\vec{r}-\vec{r}'| = s$

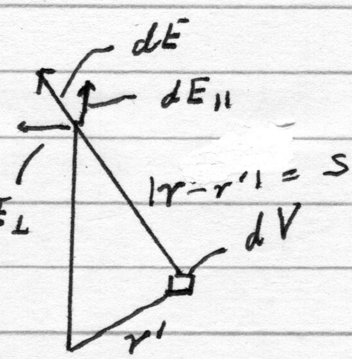
$$(1\text{-}1) \qquad \vec{E} = \int \frac{dq'}{4\pi\epsilon_0 s^2}\,\hat{r}$$

Let us make things easy for ourselves and use symmetry here

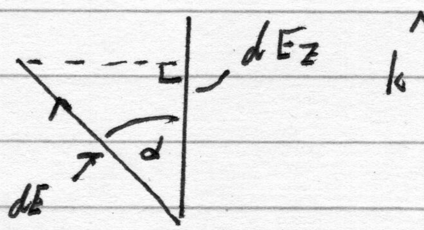
$$(1\text{-}2) \qquad \vec{E} = \int d\vec{E}$$

When we look at P, clearly $d\vec{E}$ can be broken down into two components (or $\hat{dE}$ (simply))

(1-3) $$dE = dE_\perp + dE_z$$



All of the $dE_\perp$ components add up to zero by symmetry once you integrate over the entire sphere so only the dE_z component remains, where



and

(1-4) $$dE_z = dE \cos\alpha$$

Thus we need to find

(1-5) $$E_z = \int dE_z = \int \cos\alpha \, dE$$

Or

(1-6) $$E_z = \int dE \cos\alpha = \int \frac{dq'}{4\pi\epsilon_0 S^2} \cos\alpha$$

Or

(1-7) $$E_z = \rho' \int_V \frac{dV' \cos\alpha}{4\pi\epsilon_0 S^2}$$

In spherical polar coordinates

(1-8) $$dV' = (r')^2 dr' \sin\theta' d\theta' d\phi'$$

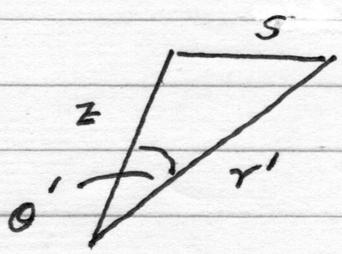
and

(1-9) $$E_z = \rho \int_V \frac{(r')^2 dr' \sin\theta' d\theta' d\phi' \cos\alpha}{4\pi\epsilon_0 S^2}$$

Where we have set $\rho = \rho'$ [N.B. This is a volume charge density and <u>not</u> a distance as in cylindrical polar coordinates!]

Now this integral depends on α, θ', ϕ', r and r'. We shall write it in terms of r, r', θ', and ϕ' only. Note also that z is fixed in this problem.

The Law of Cosines helps us out here



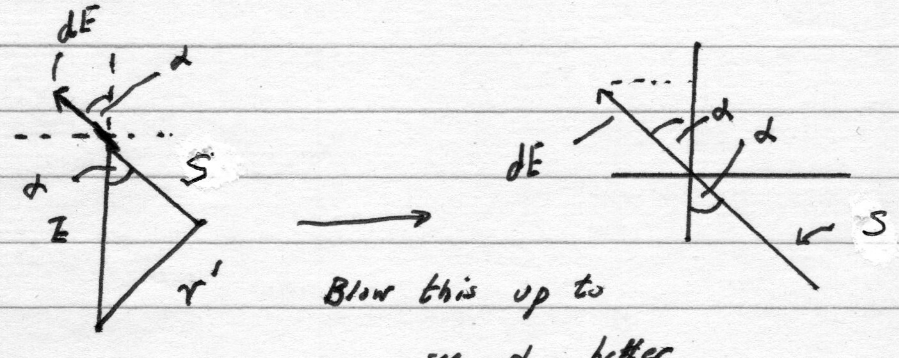
(1-10) $$s^2 = z^2 + (r')^2 - 2zr'\cos\theta'$$

(1-11) $$2r'z\cos\theta' = z^2 + (r')^2 - s^2$$

(1-12) $$\cos\theta' = \frac{[z^2 + (r')^2 - s^2]}{2zr'}$$

and we have a way to express $\theta' = \theta'(r, r')$.

Now let us express $\alpha = \alpha(r, r')$ again using the Law of Cosines



Clearly

(1-13) $$(r')^2 = z^2 + s^2 - 2zs\cos\alpha$$

or

$$(1\text{-}14) \qquad \cos\alpha = \frac{z^2 + s^2 - (r')^2}{2sz}$$

Thus by placing (1-14) in (1-9) we have eliminated α

$$(1\text{-}15) \qquad E_z = \frac{\rho}{4\pi\epsilon_0} \int_V \frac{(r')^2 dr' \sin\theta' d\theta' d\phi'}{s^2} \frac{[z^2 + s^2 - (r')^2]}{2sz}$$

It is easier to do the integral over the azimuthal angle first

$$\boxed{(1\text{-}16) \qquad E_z = \frac{\rho}{2\epsilon_0} \iint \frac{(r')^2 dr' \sin\theta' d\theta'}{s^2} \frac{[z^2 + s^2 - (r')^2]}{2sz}}$$

Now let us tackle the term $\sin\theta' d\theta'$. When we integrate over the polar angle θ' we realize from our original figure that r' is fixed, but s is not, as it is a variable. Note that we treat z as a constant when we do the integral over θ' since we are evaluating $\vec{E}(z)$ for fixed z.

Let us apply these three facts to Eq. (1-12)

(1-17) $$\cos\theta' = \frac{[z^2 + (r')^2 - s^2]}{2zr'}$$

(1-18) $$-\sin\theta' \, d\theta' = \frac{-2s\,ds}{2zr'}$$

(1-19) $$\sin\theta' \, d\theta' = \frac{s\,ds}{zr'}$$

and Eq. (1-16) becomes

(1-20) $$E_z = \frac{\rho}{2\epsilon_0} \iint \frac{(r')^2 \, dr' \, s \, ds}{zr' \, s^2} \, \frac{[z^2 + s^2 - (r')^2]}{2sz}$$

or

(1-21) $$E_z = \frac{\rho}{4\epsilon_0} \frac{1}{z^2} \iint \frac{(r')^2}{s^2 r'} \, s^2 \left[1 + \frac{z^2 - (r')^2}{s^2}\right]$$

$$\times \, ds \, dr'$$

or

(1-22) $$E_z = \frac{\rho}{4\ \epsilon_0} \frac{1}{z^2} \iint r' \left[1 + \frac{z^2 - (r')^2}{\ } \right] d\ \ dr'$$

Next let us integrate the integrand over r

(1-23) $$\int r' \left[1 + \frac{z^2 - (r')^2}{\ } \right] d$$

To find the limits of integrations, let us consult (1-10)

(1-24) $$\ ^2 = z^2 + (r')^2 - 2zr'\cos\theta'$$

When $\theta' = 0$ $\quad \ = (z - r')$ (1-24)

and

When $\theta' = \pi$ $\quad \ = (z + r')$ (1-25) so our limits of integration are for (1-23)

$$(1\text{-}26)\quad \int_{r=(z-r')}^{r=(z+r')} r'\left[1 + \frac{z^2-(r')^2}{r^2}\right] dr$$

$$(1\text{-}27)\quad \int_{r=(z-r')}^{r=(z+r')} r'\,dr + r'\left(z^2-(r')^2\right)\int_{r=(z-r')}^{r=(z+r')} r^{-2}\,dr$$

$$(1\text{-}28)\quad r'\left[z+r'-(z-r') + \left(z^2-(r')^2\right)\frac{r^{-1}}{(-1)}\Bigg|_{r=z-r'}^{r=z+r'}\right]$$

$$(1\text{-}29)\quad \left[2r' - (z^2-(r')^2)\left[\frac{1}{z+r'} - \frac{1}{z-r'}\right]\right]r'$$

$$(1\text{-}30)\quad r'\left\{2r' - \left[\left(z^2-(r')^2\right)\frac{[z-r'-z-r']}{[z^2-(r')^2]}\right]\right\}$$

or

$$(1\text{-}31)\quad r'(2r' + 2r') = 4(r')^2$$

Finally inserting (1-31) into (1-26) allows us to evaluate (1-22)

(1-32) $$E_z = \frac{\rho}{4\epsilon_0 z^2} \iint r' \left[1 - \frac{z^2 - (r')^2}{s^2}\right] dr\, dr'$$

or

(1-33) $$E_z = \frac{\rho}{4\epsilon_0 z^2} \int_{r'=0}^{r'=a} 4 (r')^2 dr'$$

or

(1-34) $$E_z = \frac{\rho}{\epsilon_0 z^2} \left.\frac{(r')^3}{3}\right|_0^a = \frac{\rho a^3}{3\epsilon_0 z^2}$$

Since $\rho = \dfrac{q}{\frac{4}{3}\pi a^3} = \dfrac{3q}{4\pi a^3}$

(1-35) $$E_z = \frac{3q\, a^3}{4\pi a^3 \; 3\epsilon_0 z^2} = \frac{q}{4\pi\epsilon_0 z^2}$$

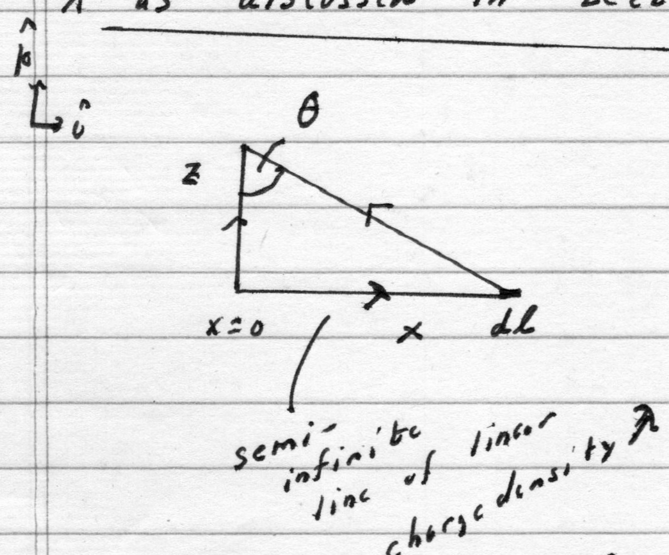
or

(1-36) $$\vec{E} = \frac{q}{4\pi\epsilon_0 z^2}\hat{k} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

Which says the spherical charge distribution of total charge q acts like a point charge as far as the electric field is concerned.

Problem 2

In Lecture 6 we calculated the electric field $\vec{E}$ at a point $\vec{P}$ above the end of a half-infinite line of linear charge density λ. We discovered the remarkable observation that $\vec{E}$ is always pointed up at an angle of 45° independent of the value of z. Repeat this calculation and then look at the same problem for the case of the other possible half-infinite line of linear charge density λ. Use these two separate results and the principle of superposition to get the expected result for the infinite line of linear charge density λ as discussed in Lecture 5.



$$dq = \lambda\, dl$$

$$\vec{r} - \vec{r}\,' = -\hat{i}x + \hat{k}z$$

$$\vec{r}\,' = \hat{i}x$$

$$\vec{r} = \hat{k}z$$

$$|\vec{r} - \vec{r}\,'| = (x^2 + z^2)^{1/2}$$

$$\vec{E} = \int \frac{dq}{4\pi\epsilon_0} \frac{\hat{r}}{|\vec{r} - \vec{r}\,'|^2} = \int \frac{dq\,(\vec{r} - \vec{r}\,')}{4\pi\epsilon_0 |\vec{r} - \vec{r}\,'|^3}$$

From the figure

$$\tan\theta = \frac{x}{z} \qquad x = z\tan\theta$$

$$dx = z\sec^2\theta\, d\theta$$

$$1 + \tan^2\theta = \sec^2\theta = 1 + \frac{x^2}{z^2}$$

$$\vec{E} = \int \frac{\lambda\, dl\, (-\hat{\imath}x + \hat{k}z)}{4\pi\epsilon_0\, (x^2+z^2)^{3/2}}$$, where z is fixed

Let us do the improper integral first and then take limit as $L \to \infty$

Using our trigonometry from above

$$[x^2+z^2]^{3/2} = z^3\left[1 + \left(\frac{x}{z}\right)^2\right]^{3/2} = z^3\sec^3\theta$$

$$\vec{E}_1 = \frac{-\hat{\imath}\lambda}{4\pi\epsilon_0 z^3}\int_0^L \frac{z\tan\theta\; z\sec^2\theta\, d\theta}{\sec^3\theta}$$

$$\vec{E}_1 = \frac{-\hat{\imath}\lambda}{4\pi\epsilon_0 z}\int_0^L \frac{\sin\theta}{\cos\theta}\cos\theta\, d\theta$$

$$\vec{E}_1 = \frac{-\hat{i}\lambda}{4\pi\epsilon_0 z}\int_0^L \sin\theta\, d\theta =$$

$$\frac{+\hat{i}\lambda}{4\pi\epsilon_0 z}\cos\theta\Big|_0^L$$

From diagram $\cos\theta = \frac{z}{(x^2+z^2)^{1/2}}$

$$\vec{E} = \frac{+\hat{i}\lambda}{4\pi\epsilon_0 z}\frac{z}{(x^2+z^2)^{1/2}}\Big|_0^L$$

$$\vec{E} = \frac{\hat{i}\lambda}{4\pi\epsilon_0 z}\left[\frac{z}{(L^2+z^2)^{1/2}} - 1\right]$$

As $L \to \infty$ $\quad \boxed{\vec{E}_1 \to \frac{-\hat{i}\lambda}{4\pi\epsilon_0 z}}$

Now what about

$$\vec{E}_2 = \frac{\hat{k}\lambda z}{4\pi\epsilon_0}\int_0^L \frac{dx}{[x^2+z^2]^{3/2}}$$

Again for $\tan\theta = \frac{x}{z}$, for fixed z, we use trigonometric substitution to find

$$\vec{E}_2 = \frac{\hat{k}\lambda z}{4\pi\epsilon_0}\int\frac{z\sec^2\theta\, d\theta}{z^3\sec^3\theta} = \frac{\hat{k}\lambda z^2}{4\pi\epsilon_0}\frac{1}{z^3}\int\cos\theta\, d\theta$$

$$\vec{E}_2 = \frac{\lambda \hat{k} \sin\theta}{4\pi\epsilon_0 z} \doteq \frac{\hat{k}\lambda}{4\pi\epsilon_0 z} \frac{x}{(x^2+z^2)^{1/2}} \Bigg|_0^L$$

(follows from figure)

$$\vec{E}_2 = \frac{\hat{k}\lambda}{4\pi\epsilon_0 z} \frac{L}{(L^2+z^2)^{1/2}} = \frac{\hat{k}\lambda}{4\pi\epsilon_0 \left[1 + \frac{z^2}{L^2}\right]^{1/2} z}$$

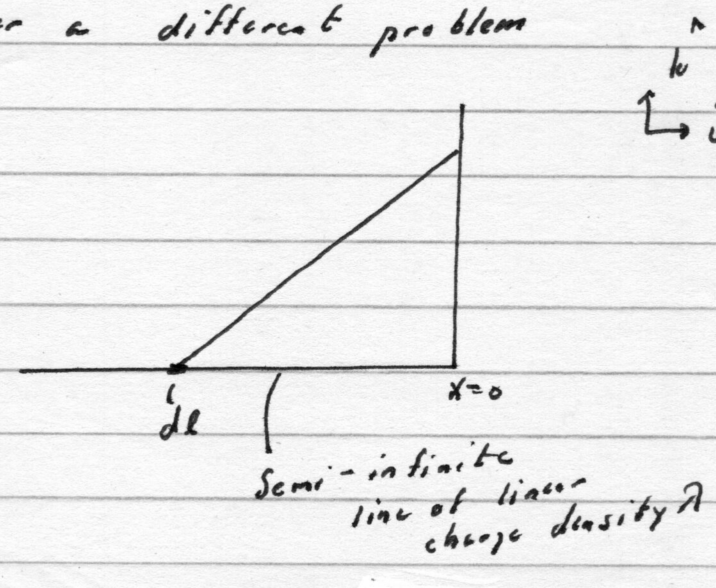
As $L \to \infty$ $\quad \vec{E}_2 \to \dfrac{\hat{k}\lambda}{4\pi\epsilon_0 z}$

For a semi-infinite line from $0 \le x < \infty$

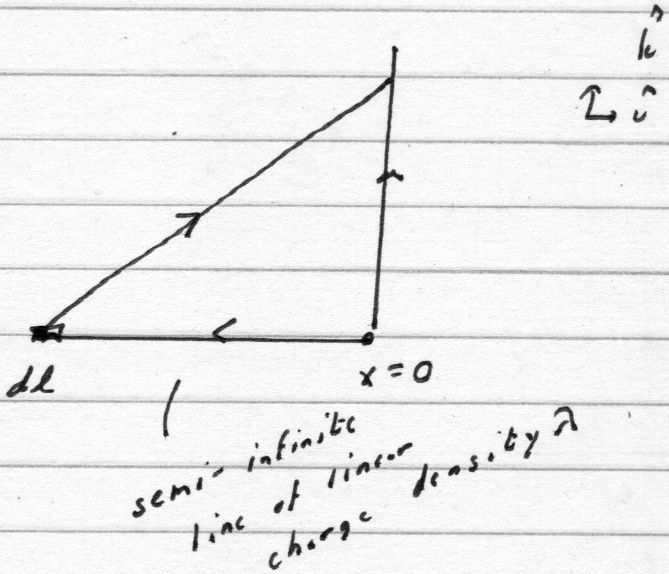
$$\boxed{\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{-\hat{i}\lambda}{4\pi\epsilon_0 z} + \frac{\hat{k}\lambda}{4\pi\epsilon_0 z}}$$

as we saw in lecture.

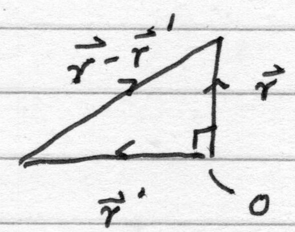
Now consider a different problem



In this case



In this case



$$\vec{r} = \hat{k}\, z$$
$$\vec{r}' = -\hat{\imath}\, x$$
$$\vec{r} - \vec{r}' = \hat{\imath}\, x + \hat{k}\, z$$
$$|\vec{r} - \vec{r}'| = (x^2 + z^2)^{1/2}$$

$$\vec{r}' - \vec{r}' = \hat{i}x + \hat{k}Z$$

$$|\vec{r} - \vec{r}'| = (x^2 + z^2)^{1/2}$$

$$\vec{E} = \int \frac{dq}{4\pi\epsilon_0} \frac{\hat{r}}{|\vec{r} - \vec{r}'|^2} = \int \frac{dq\,(\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3}$$

$$\vec{E} = \frac{\hat{i}\lambda}{4\pi\epsilon_0} \int \frac{x\,dx}{[x^2 + z^2]^{3/2}} + \int \frac{\hat{k}\lambda\, z\,dx}{[x^2 + z^2]^{3/2}}$$

We have done the first integral before, but now the limits of integration are different

$$\vec{E}_1 = \frac{+\hat{i}\lambda}{4\pi\epsilon_0} \int_{-L}^{0} \frac{x\,dx}{[x^2 + z^2]^{3/2}} = \frac{+\hat{i}\lambda}{4\pi\epsilon_0 Z} \frac{Z}{(x^2 + Z^2)^{1/2}} \Bigg|_{-L}^{0}$$

$$\vec{E}_1 = \frac{\hat{i}\lambda}{4\pi\epsilon_0} \left[1 - \frac{Z}{Z(L^2 + Z^2)^{1/2}} \right]$$

As $-L \to -\infty$ $\qquad \vec{E}_1 \to \dfrac{\hat{i}\lambda}{4\pi\epsilon_0 Z}$

Similarly for $\vec{E}_2$

$$\vec{E}_2 = \frac{\lambda \hat{k} z}{4\pi\epsilon_0} \int_{-L}^{0} \frac{dx}{[x^2+z^2]^{3/2}}$$

$$\vec{E}_2 = \frac{\lambda \hat{k}}{4\pi\epsilon_0} \frac{1}{z} \left[\frac{x}{(x^2+z^2)^{1/2}} \right]_{-L}^{0}$$

$$\vec{E}_2 = \frac{\lambda \hat{k}}{4\pi\epsilon_0} \frac{1}{z} \left[\frac{+L}{(L^2+z^2)^{1/2}} \right]$$

Note that

$$(L^2+z^2)^{1/2} = L\left(1+\left(\frac{z}{L}\right)^2\right)^{1/2}$$

and as $L \to \infty$ this goes to L

or

$$\vec{E}_2 = \frac{\lambda \hat{k}}{4\pi\epsilon_0 z}$$

Thus for our different problem

$$\boxed{\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{+\hat{i}\lambda}{4\pi\epsilon_0} + \frac{\hat{k}\lambda}{4\pi\epsilon_0 z}}$$

Combining the two results for each of the semi-infinite lines gives

The principle of superposition works!!!

$$\vec{E} = \frac{\hat{k}\lambda}{2\pi\epsilon_0 z}$$

which is what we expect from Lecture 5 for the infinite wire case!

Problem 3

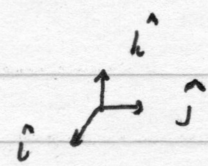
In Lecture 6 we calculated the electric field $\vec{E}$ of a thin plastic rod bent into a semicircle of radius a with a linear charge density

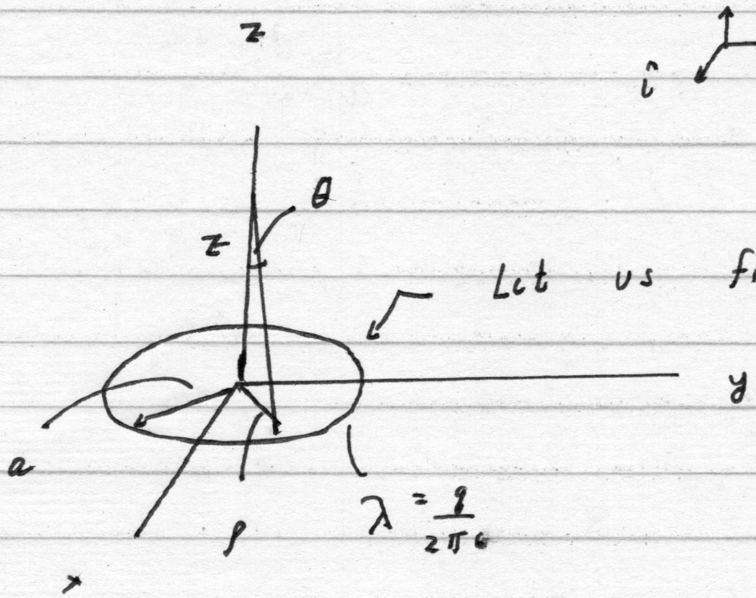
$$\lambda = \frac{q}{2\pi a}$$

We found $\vec{E}$ at the center of the circle. Repeat this calculation and then look at the same problem for the case of another thin plastic rod bent into the other semicircle of radius a with a linear charge density

$$\lambda = \frac{q}{2\pi a}$$

Use these two separate results and the principle of superposition to get the expected result for the circular thin rod of linear charge density λ as seen in Lecture 5.





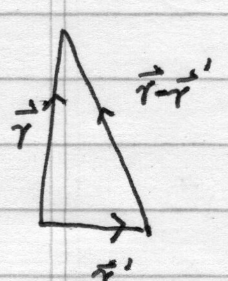
Let us first set up the circular ring case for simplicity!

$$\rho = a \quad [\text{constant}]$$

$$dq = \lambda\, dl \quad = \quad \lambda \rho\, d\phi \quad [\text{in plane polar coordinates}]$$

$$\vec{E} = \int \frac{dq\ \hat{r}}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|^2} = \vec{E} = \int \frac{dq\ (\vec{r}-\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|^3} = \int \frac{\lambda \rho\, d\phi\ (\vec{r}-\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|^3}$$

In this problem



$$\vec{r} = \vec{z}$$

$$\vec{r}' = \vec{\rho}$$

$$\vec{r}-\vec{r}' = \vec{z} - \vec{\rho} = -\rho[\hat{i}\cos\phi + \hat{j}\sin\phi] + \hat{k}\, z$$

or

$$\vec{r}-\vec{r}' = -a\,[\hat{i}\cos\phi + \hat{j}\sin\phi] + \hat{k}\, z$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 = \lambda \int \frac{a\,d\phi\,(-\hat{i}\cos\phi)\,a}{4\pi\epsilon_0 [a^2+z^2]^{3/2}}$$

$$+ \frac{\lambda}{4\pi\epsilon_0} \int \frac{a^2\,d\phi\,(-\hat{j}\sin\phi)}{[a^2+z^2]^{3/2}} + \frac{\lambda a}{4\pi\epsilon_0} \int \frac{\hat{k}\,z\,d\phi}{[a^2+z^2]^{3/2}}$$

Let us tackle the first integral

$$\vec{E}_1 = \frac{-\hat{i}\lambda a^2}{4\pi\epsilon_0} \int \frac{1}{[a^2+z^2]^{3/2}} \cos\phi\,d\phi$$

Note that since a is a constant and z is fixed

$$\vec{E}_1 = \frac{-\hat{i}\lambda a^2}{4\pi\epsilon_0} \frac{1}{[a^2+z^2]^{3/2}} \int \cos\phi\,d\phi$$

Now let us consider a semi-circle so $0 \le \phi \le \pi$ only! Clearly

$$\int_0^\pi \cos\phi\,d\phi = -\sin\phi \Big|_0^\pi \to 0$$

so

$$\vec{E}_1 = 0$$

What about $\vec{E}_2$?

$$\vec{E}_2 = \frac{-\hat{j}\,\lambda a^2}{4\pi\epsilon_0}\frac{1}{[a^2+z^2]^{3/2}}\int d\phi \sin\phi$$

Again $0 \leq \phi \leq \pi$ for our semi-circle so

$$\vec{E}_2 = \frac{-\hat{j}\,\lambda a^2}{4\pi\epsilon_0}\frac{1}{[a^2+z^2]^{3/2}}\Big[-\cos\phi\Big]_0^{\pi}$$

$$\vec{E}_2 = [+1+1]\frac{(-\hat{j}\,\lambda a^2)}{4\pi\epsilon_0\,[a^2+z^2]^{3/2}}$$

$$\vec{E}_2 = \frac{-2\lambda a^2\hat{j}}{4\pi\epsilon_0\,[a^2+z^2]^{3/2}}$$

What about $\vec{E}_3$?

$$\vec{E}_3 = \frac{\lambda a\, z\hat{k}}{4\pi\epsilon_0}\int\frac{d\phi}{[a^2+z^2]^{3/2}}$$

$$\vec{E}_3 = \frac{\lambda a z\hat{k}}{4\pi\epsilon_0}\frac{1}{[a^2+z^2]^{3/2}}\int_0^{\pi} d\phi$$

$$\vec{E}_3 = \frac{\lambda a z \hat{k}}{4\epsilon_0 [a^2+z^2]^{3/2}}$$

since we are dealing with a semi-circle here

In summary,

$$\boxed{\vec{E} = \vec{E}_2 + \vec{E}_3 = \frac{\lambda a z \hat{k}}{4\epsilon_0 [a^2+z^2]^{3/2}} + \frac{(-2\lambda a^2 \hat{j})}{4\pi\epsilon_0 [a^2+z^2]^{3/2}}}$$

Now if we look at the other semi-circle nothing changes except $\pi \le \phi \le 2\pi$

$$\vec{E}_1 = \frac{-\hat{i}\lambda a^2}{4\pi\epsilon_0} \frac{1}{[a^2+z^2]^{3/2}} \int_{\pi}^{2\pi} \cos\phi \, d\phi$$

and since

$$\int_{\pi}^{2\pi} \cos\phi \, d\phi = \sin\phi \Big|_{\pi}^{2\pi} = 0$$

$$\vec{E}_1 = 0$$

$$\vec{E}_2 = -\frac{\hat{j}\lambda a^2}{4\pi\epsilon_0} \frac{1}{[a^2+z^2]^{3/2}} \int_{\pi}^{2\pi} d\phi \sin\phi$$

and

since $\int_{\pi}^{2\pi} d\phi \sin\phi =$

$$-\cos\phi\Big|_{\pi}^{2\pi} =$$

$$(-)(1) + [(-1)]$$

$$= -2$$

$$\vec{E}_2 = \frac{+\hat{j}\lambda a^2}{2\pi\epsilon_0} \frac{1}{[a^2+z^2]^{3/2}}$$

or

$$\vec{E}_2 = \frac{2\hat{j}a^2\lambda}{4\pi\epsilon_0 [a^2+z^2]^{3/2}}$$

Finally what about $\vec{E}_3$?

$$\vec{E}_3 = \frac{\lambda a z \hat{k}}{4\pi\epsilon_0} \frac{1}{[a^2+z^2]^{3/2}} \int_{\pi}^{2\pi} d\phi$$

$$\vec{E}_3 = \frac{\lambda a z \hat{k}}{4\epsilon_0} \frac{1}{[a^2+z^2]^{3/2}}$$

or for this semi-circle

$$\vec{E} = \vec{E}_2 + \vec{E}_3$$

$$\boxed{\vec{E} = \frac{2a^2\lambda\hat{j}}{4\pi\epsilon_0[a^2+z^2]^{3/2}} + \frac{\lambda a z \hat{k}}{4\epsilon_0[a^2+z^2]^{3/2}}}$$

Now the principle of superposition allows us to add these two results of each semi-circle to get a full circle

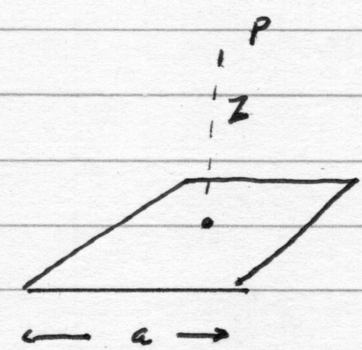
$$\vec{E}_{\text{full circle}} = \frac{\lambda a z \hat{k}}{2\epsilon_0[a^2+z^2]^{3/2}}$$

which is exactly what we discovered in lecture 5 !!!

Problem 4

Show that the electric field $\vec{E}$ at a distance z above the center of a square loop of side a carrying a uniform linear charge density λ is

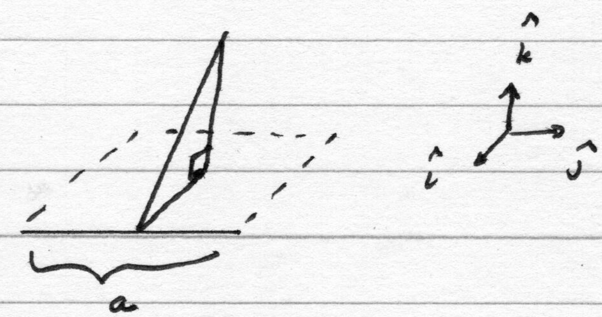
$$\vec{E}(z) = \frac{\lambda a z \, \hat{k}}{\pi \epsilon_0 \left[z^2 + \frac{a^2}{4}\right]\left[z^2 + \frac{a^2}{2}\right]^{1/2}}$$



Problem 6 in Problem Set V tells us that the $\vec{E}$ field for a finite charged wire of length 2L and linear charge density λ is

$$\vec{E} = \frac{\lambda 2L \, \hat{k}}{4\pi \epsilon_0 z [z^2 + L^2]^{1/2}}$$

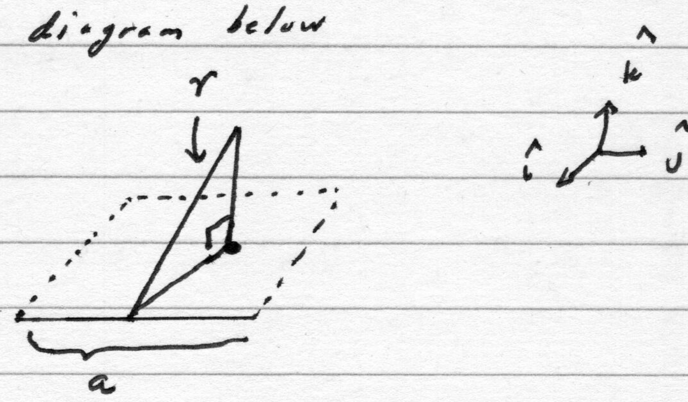
Let us try to take advantage of this expression and focus on a single side of our square loop of length a



Clearly we can replace 2L by a in our original expression to get

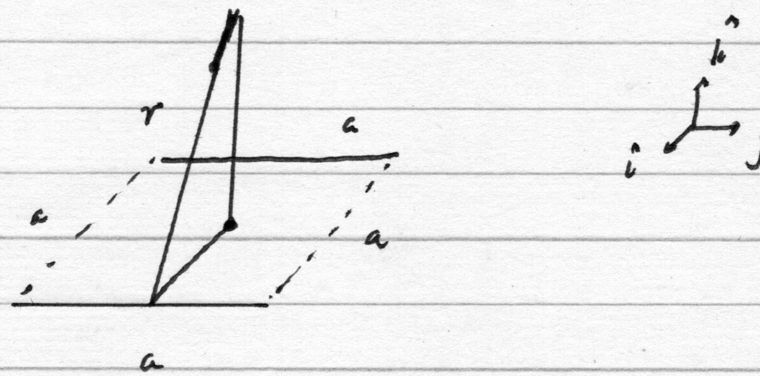
$$\vec{E} = \frac{\lambda a \hat{k}}{4\pi\epsilon_0 z\left[z^2 + \frac{a^2}{4}\right]^{1/2}}$$

The next term to replace is z which we will call r as it is clearly not z in our diagram below

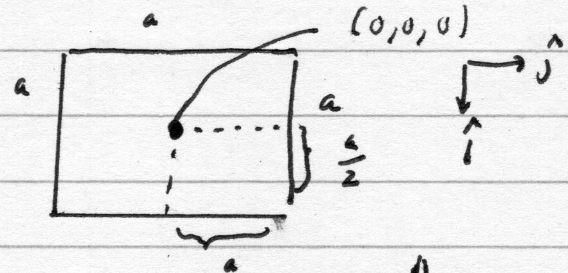


$$\vec{E} = \frac{\lambda a \hat{k}}{4\pi\epsilon_0 r\left[r^2 + \frac{a^2}{4}\right]^{1/2}}$$

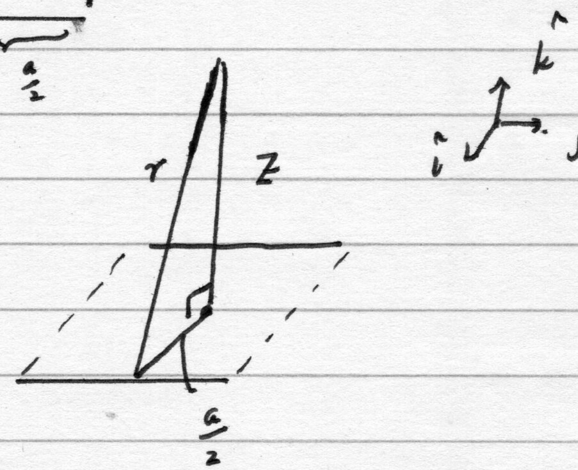
Now what exactly is r in terms of the important parameters of our problem?



Well if we look at the above figure from the perspective along the $\hat{k}$ axis



then clearly



and

$$r = \left(Z^2 + \frac{a^2}{4}\right)^{1/2}$$

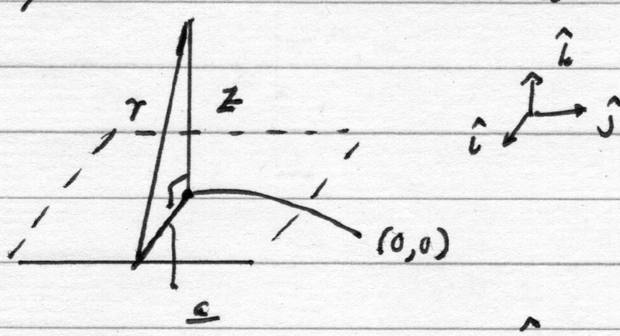
so that we now have

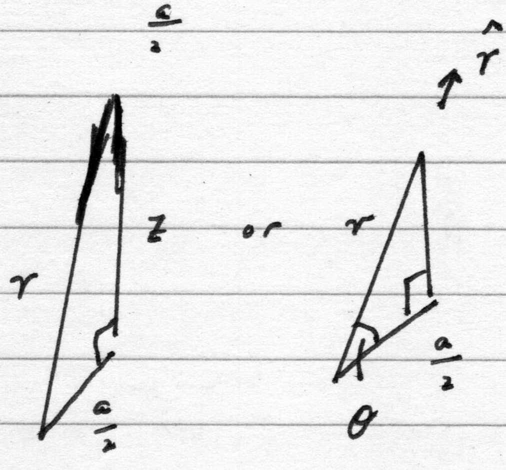
$$\vec{E} = \frac{\lambda a \hat{k}}{4\pi\epsilon_0 \left[z^2 + \frac{a^2}{4}\right]^{1/2} \left[z^2 + \frac{a^2}{4} + \frac{a^2}{4}\right]^{1/2}}$$

or

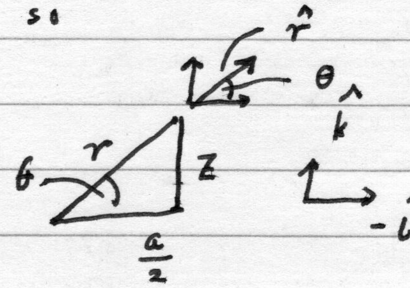
$$\vec{E} = \frac{\lambda a \hat{k}}{4\pi\epsilon_0 \left[z^2 + \frac{a^2}{4}\right]^{1/2} \left[z^2 + \frac{a^2}{2}\right]^{1/2}}$$

Now this is almost correct. We next need to replace $\hat{k}$ by the appropriate unit vector for our problem. From the figure below let us extract the triangle

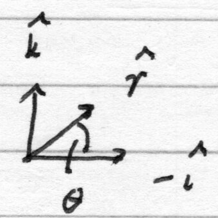




What is $\hat{r}$?
It lies in xz-plane
so

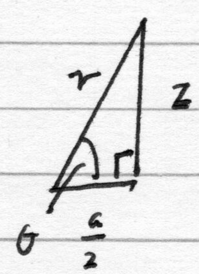


and we can write $\hat{r}$ in terms of its components



$$\hat{r} = -\hat{i}\cos\theta + \hat{k}\sin\theta$$

From our triangle we see



or $\sin\theta = \dfrac{z}{r}$

$\cos\theta = \dfrac{a}{2r}$

or

$$\cos\theta = \frac{a}{2\left[z^2 + \frac{a^2}{4}\right]^{1/2}}$$

$$\sin\theta = \frac{z}{\left[z^2 + \frac{a^2}{4}\right]^{1/2}}$$

so we are finished as far as one side is concerned. Let us put all of this together

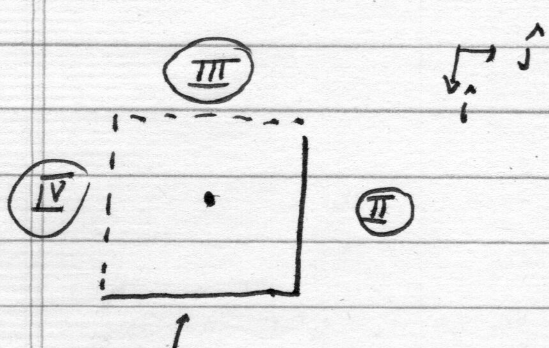
$$\vec{E} = \frac{\lambda a \left[-\hat{i} \cos\theta + \hat{k} \sin\theta \right]}{4\pi\epsilon_0 \left[z^2 + \frac{a^2}{4} \right]^{1/2} \left[z^2 + \frac{a^2}{2} \right]^{1/2}}$$

or

$$\vec{E} = \frac{-\lambda a \hat{i}}{4\pi\epsilon_0} \frac{a}{2\left[z^2 + \frac{a^2}{4} \right] \left[z^2 + \frac{a^2}{2} \right]^{1/2}}$$

$$+ \frac{\lambda a \hat{k}}{4\pi\epsilon_0} \frac{z}{\left[z^2 + \frac{a^2}{4} \right] \left[z^2 + \frac{a^2}{2} \right]^{1/2}}$$

This is the electric field just from one side



SIDE I

where if we let

$$\eta = \frac{1}{\left[z^2 + \frac{a^2}{4} \right] \left[z^2 + \frac{a^2}{2} \right]^{1/2}}$$

$$\vec{E}_I = \frac{-\lambda a \hat{i}}{4\pi\epsilon_0} \frac{a\eta}{2} + \frac{\lambda a \hat{k} z}{4\pi\epsilon_0} \eta$$

Clearly the other three sides have contributions

$$E_{II} = \frac{-\hat{j}\lambda a}{4\pi\epsilon_0} \frac{a\eta}{2} + \frac{\lambda a \hat{k} z}{4\pi\epsilon_0}\eta$$

$$E_{III} = \frac{+\hat{i} a \lambda}{4\pi\epsilon_0} \frac{a\eta}{2} + \frac{\lambda a \hat{k} z\eta}{4\pi\epsilon_0}$$

$$E_{IV} = \frac{\hat{j}\lambda a}{4\pi\epsilon_0} \frac{a\eta}{2} + \frac{\lambda a \hat{k} z\eta}{4\pi\epsilon_0}$$

Finally by using the principle of superposition all four of these terms add up to

$$\vec{E} = \frac{4\lambda a \hat{k} z\eta}{4\pi\epsilon_0}$$

or

$$\vec{E} = \frac{4\lambda a z\hat{k}}{4\pi\epsilon_0} \frac{1}{\left[z^2+\frac{a^2}{4}\right]\left[z^2+\frac{a^2}{2}\right]^{1/2}}$$

N.B. You could have obtained this result a bit faster using symmetry!

$$\boxed{\vec{E} = \frac{\lambda a z\hat{k}}{\pi\epsilon_0} \frac{1}{\left[z^2+\frac{a^2}{4}\right]\left[z^2+\frac{a^2}{2}\right]^{1/2}}}$$

This is wonderful!

Problem 5

Show that the curl of the gradient of a scalar field vanishes.

Let V be a scalar field, $V(x, y, z)$

$$\vec{\nabla}(V) = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$$

$$\vec{\nabla} \times (\vec{\nabla} V) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} =$$

$$\text{curl }(\vec{\nabla} V) = \hat{i}\left[\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y}\right] + \hat{j}\left[\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z}\right]$$

$$+ \hat{k}\left[\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right]$$

This vanishes provided $\boxed{(\text{curl grad } V = 0)}$

$$\frac{\partial}{\partial x}\left[\frac{\partial V}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\partial V}{\partial x}\right] \; ; \; \frac{\partial}{\partial y}\left[\frac{\partial V}{\partial z}\right] = \frac{\partial}{\partial z}\left[\frac{\partial V}{\partial y}\right]$$

$$\frac{\partial}{\partial z}\left[\frac{\partial V}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial V}{\partial z}\right]$$

which is true provided the mixed partial derivatives are continuous.

In summary $\quad \text{curl grad } V = 0$

Problem 6

Show that the divergence of the curl of a vector field vanishes.

Let $\vec{A}$ be a vector field

$$\vec{\nabla} \times \vec{A} = \hat{i}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \hat{j}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]$$

$$+ \hat{k}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \frac{\partial}{\partial x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]$$

$$+ \frac{\partial}{\partial y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z}$$

$$+ \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

or

$$\text{div } (\text{cur } \vec{A}) = \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_z}{\partial y \partial x}$$

$$+ \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_x}{\partial z \partial y}$$

Again assuming that mixed partial derivatives are equivalent as in the previous problem

$$\text{div } (\text{curl } \vec{A}) = 0$$

Problem 7

See if you can express the divergence of the gradient in a fairly simple form.

Let f be a scalar field, then

$$\vec{\nabla}\cdot(\vec{\nabla} f) = \text{div grad} f$$

$$\vec{\nabla}\cdot\left[\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right]$$

$$= \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right]$$

or

$$\vec{\nabla}\cdot(\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\equiv \nabla^2 f$$

where $\boxed{\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$

is defined as the Laplacian operator. We will come back to a discussion of this in a later problem set.