USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
Dr. Steven L. Richardson (srichards22@comcast.net)
Professor Emeritus of Electrical Engineering, Department of Electrical and Computer Engineering, Howard University, Washington, DC
and
Faculty Associate in Applied Physics, John A. Paulson School of Engineering and Applied Sciences, Harvard University, Cambridge, MA

PROBLEM SET V
(due Tuesday, July 28, 2020)

Problem 1

Show that in cylindrical polar coordinates the divergence of a vector field $\vec{A}$ is

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

You should start with

$$\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}$$

and use the relevant information from Problem Set I to recast the unit vectors from Cartesian coordinates to cylindrical polar coordinates in the above expression. Next starting with the expression for the divergence of $\vec{A}$ in Cartesian coordinates and the relevant partial derivatives from Problem Set IV, proceed carefully and boldly from there. You should note that this process is straightforward and it also applies to the following Problem 2 and Problem 3. It is, of course, tedious, and there are more sophisticated ways of obtaining the desired result, but you will nevertheless learn a lot from doing this in a tedious manner (e.g. patience, correcting your mistakes, organizational skills in solving a problem, partial differentiation, etc.).

Problem 2

Show that in spherical polar coordinates the divergence of a vector field $\vec{A}$ is

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

You should start with

$$\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}$$

and use the relevant information from Problem Set I to recast the unit vectors from Cartesian coordinates to spherical polar coordinates in the above expression. Next starting with the expression for the divergence of $\vec{A}$ in Cartesian coordinates and the relevant partial derivatives from Problem Set IV, proceed carefully and boldly from there. You should note that this process is straightforward and it also applies to both Problem 1 and Problem 3. It is, of course, tedious, and there are more sophisticated ways of obtaining the desired result, but you will nevertheless learn a lot from doing this in a tedious manner (e.g. patience, correcting your mistakes, organizational skills in solving a problem, partial differentiation, etc.).

Problem 3

Show that in cylindrical polar coordinates the curl of a vector field $\vec{A}$ is

$$\nabla \times \vec{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\rho} + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\hat{k}$$

You should start with

$$\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}$$

and use the relevant information from Problem Set I to recast the unit vectors from Cartesian coordinates to spherical polar coordinates in the above expression. Next starting with the expression for the divergence of $\vec{A}$ in Cartesian coordinates and the relevant partial derivatives from Problem Set IV, proceed carefully and boldly from there. You should note that this process is straightforward and it also applies to Problem 1 and Problem 2. It is, of course, tedious, and there are more sophisticated ways of obtaining the desired result, but you will nevertheless learn a lot from doing this in a tedious manner (e.g. patience, correcting your mistakes, organizational skills in solving a problem, partial differentiation, etc.). I should note that of the three problems given so far, this one is the most challenging.

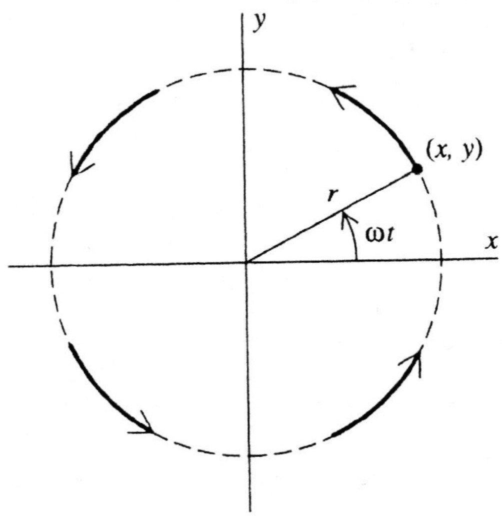
Problem 4

Calculating the curl of a vector field is one thing. Understanding what it means physically is quite another. Let us look at a few examples from the motion of fluids. Suppose water is draining from a bathtub. A small of volume of the water at a point (x, y) at time t has the following coordinates

$$x = r \sin \omega t$$

$$y = r \cos \omega t$$

where ω is the constant velocity of the water as shown in the figure below.



We should say that this is not a realistic description of water draining from a tub since rotating water shears tangentially and its angular velocity will therefore vary with r. The crude description we use here is adequate for our purposes and has the virtue of being simple. Show that its velocity at (x, y) is given by the velocity field of water which tells us the velocity of the water at any point (x, y).

$$\vec{v} = \omega(-\hat{\imath}y + \hat{\jmath})$$

Your intuition probably tells you that, because the motion is circular, this velocity must have a nonzero curl. In fact, please show that

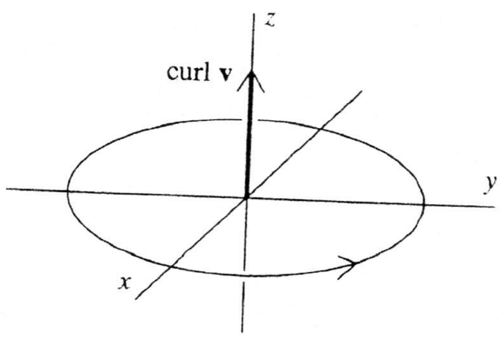
$$\nabla \times \vec{v} = 2\,\hat{k}\,\omega$$

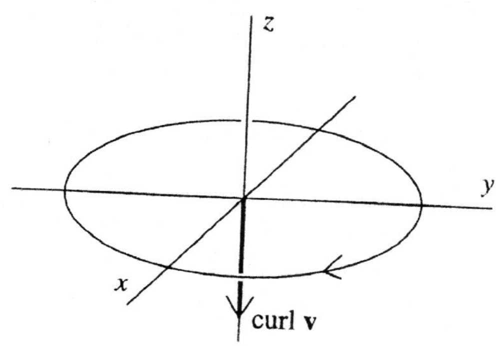
This result is quite reasonable because it says that the curl of the velocity field is proportional to the angular velocity of the swirling water. We see that

$$\nabla \times \vec{v}$$

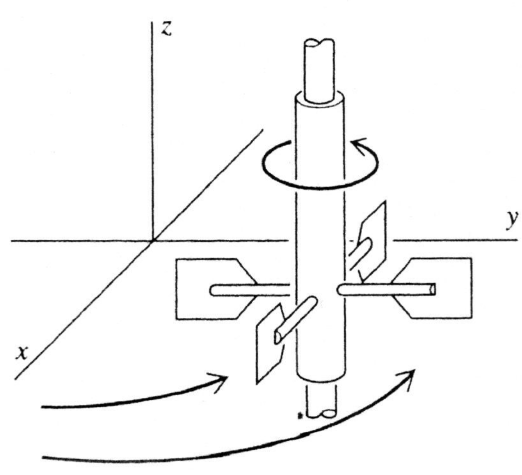
is a vector perpendicular to the plane of motion and in the positive z-direction.

If the water were rotating around in the other direction, then the curl of the velocity field would then be in the negative z-direction.





Note that this is consistent with the right-hand rule. If we were to place a small paddle wheel in the water, it would commence spinning because the impinging water would exert a net torque on the paddles. Furthermore the paddle wheel would rotate with its axis pointing in the direction of the curl.



Sketch the velocity field below, calculate its curl, and describe what a paddle wheel would do when placed in the fluid

$$\vec{v} = \hat{\jmath}\, v_o\, e^{-\frac{y^2}{\lambda^2}}$$

Now do the same for this velocity field. Be careful!

$$\vec{v} = \hat{\jmath}\, v_o\, e^{-\frac{x^2}{\lambda^2}}$$

Problem 5

In Lecture 5, we discussed the problem of calculating the electric field $\vec{E}$ at a distance z above the midpoint between two equal charges q a distance d apart. Now consider the problem of calculating the electric field $\vec{E}$ at a distance z above the midpoint between two unequal charges q and $-q$ at a distance d apart.

Problem 6

In Lecture 5, we discussed the problem of calculating the electric field $\vec{E}$ at a distance z above the midpoint of a straight line segment of infinite length that carries a uniform linear charge density λ. Now consider the problem of calculating the electric field $\vec{E}$ at a distance z above the midpoint of a straight line segment of finite length $2L$ that carries a uniform linear charge density λ.

Problem 7

Calculate the electric field $\vec{E}$ at a distance z above one end of a straight line segment of finite length L that carries a uniform linear charge density λ.

Problem 8

Find the electric field $\vec{E}$ at a distance z above the center of a flat circular disk of radius a that carries a uniform surface charge density σ. What does your formula give in the limit that $a \to \infty$? Also check the case where $z \gg a$.

Problem 9

A hole of radius a is cut out from a infinite flat sheet of uniform surface charge density σ. Let L be the line perpendicular to the sheet, passing through the center of the hole. What is the electric field $\vec{E}$ at a distance z above the center of the hole? *Hint* : Consider the infinite flat sheet to consist of many concentric rings.

Problem 10

Find the divergence of the following vector field

$$\vec{Q} = \rho\,\sin\phi\,\hat{\rho} + \rho\,z^2\,\hat{\phi} + z\,\cos\phi\,\hat{z}$$

Problem 11

Find the divergence of the following vector field at the point $(5, \frac{\pi}{2}, 1)$

$$\vec{B} = \rho\,z\sin\phi\,\hat{\rho} + 3\rho\,z^2\,\cos\theta\,\hat{\phi}$$

Problem 12

Find the divergence of the following vector field

$$\vec{G} = 10\,e^{-2z}(\rho\hat{\rho} + \hat{k})$$

Problem 13

Find the divergence of the following vector field

$$\vec{T} = \frac{1}{r^2}\cos\theta\,\hat{r} + r\,\sin\theta\cos\phi\,\hat{\theta} + \cos\theta\,\hat{\phi}$$

Problem 14

Find the divergence of the following vector field at the point $(1, \frac{\pi}{6}, \frac{\pi}{3})$

$$\vec{C} = 2r\,\cos\theta\cos\phi\,\hat{r} + r^{\frac{1}{2}}\,\hat{\phi}$$

Problem 15

Find the divergence of the following vector field

$$\vec{Y} = \left(r^2 + \sin\theta\,\cos\phi\right)\,\hat{r} + (r\,\sin\theta\,\cos\phi)\,\hat{\theta} + \left(\frac{1}{r}\tan\theta\, + \sin\phi\right)\,\hat{\phi}$$

Problem 16

(a) Twelve equal charges, q, are situated at the corners of a regular 12-sided polygon (for instance, one on each numeral of a clock face). What is the net force on a test charge Q at the center of the polygon?

(b) Suppose *one* of the twelve $q's$ is removed (the one at "6 o'clock"). What is the force on Q? Explain your reasoning carefully.