USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET IX
(due Tuesday, August 25, 2020)

Problem 1

In Lecture 9 we calculated the electrostatic potential V *outside* a uniformly charged solid sphere of radius a and total charge density q. We left out a few steps in our derivation and left them for this problem set.

Let us start with the step where we have done the azimuthal integral

$$V(z) = \frac{\rho}{2\,\epsilon_o} \int\int \frac{(r')^2\, dr' \,\sin\theta'\, d\theta'}{\sqrt{z^2 + (r')^2 - 2\,z\,r'\,\cos\theta'}} \tag{1}$$

Next we must perform the polar integral over θ'. We first introduce a dummy variable u where

$$u = z^2 + (r')^2 - 2\,z\,r'\,\cos\theta' \tag{2}$$

When we integrate over the polar angle θ' we realize from our original figure that r' is fixed . Note also that we treat z as a constant when we do the integral over θ' since we are evaluating the electrostatic potential $V(z)$ at a fixed value of z. Given these observations show that

(3)
$$\sin\theta' d\theta' = \frac{du}{2\, r'\, z}$$

and Eq. (1) becomes

(4)
$$V(z) = \frac{\rho}{2\,\epsilon_o} \int\int \frac{1}{2\, z\, r'} \frac{du}{\sqrt{u}} \,(r')^2\, dr'$$

Now integrate the inner integral over u and use Eq. (2) to find the appropriate limits of integration as the polar angle goes from $\theta' = 0$ to $\theta' = \pi$. You should obtain the result

(5)
$$V(z) = \left[\frac{\rho}{2\,\epsilon_o\, z} \int \sqrt{(z^2 + r')^2} - \sqrt{(z^2 - r')^2}\right] r'\, dr'$$

To evaluate this integral only take the positive square roots and be sure that you are outside the sphere. You should get a familiar result for the electrostatic potential V. Use this result to get a familiar result for the electric field $\vec{E}$

(6)
$$\vec{E} = \int \frac{q}{4\pi\epsilon_o r^2}\,\hat{r}$$

where q is the total charge of the sphere. Note that your answer is identical to the case where the total charge q is concentrated at the center of the sphere which is neat!

Problem 2

In Lecture 9 we calculated the electrostatic potential *inside* a uniformly charged solid sphere of radius a and total charge density q. We left out a few steps in our derivation and left them for this problem set.

Let us start with the step where we have done the azimuthal integral

(7)
$$V(z) = \frac{\rho}{2\,\epsilon_o} \int\int \frac{(r')^2\, dr'\, \sin\theta'\, d\theta'}{\sqrt{z^2 + (r')^2 - 2\, z\, r'\, \cos\theta'}}$$

Next we must perform the polar integral over θ'. We first introduce a dummy variable u where

(8)
$$u = z^2 + (r')^2 - 2\, z\, r'\, \cos\theta'$$

When we integrate over the polar angle θ' we realize from our original figure that r' is fixed . Note also that we treat z as a constant when we do the integral over θ' since we are evaluating the electrostatic potential $V(z)$ at a fixed value of z. Given these observations show that

(9)
$$\sin\theta' d\theta' = \frac{du}{2\, r'\, z}$$

and Eq. (7) becomes

(10)
$$V(z) = \frac{\rho}{2\,\epsilon_o} \int \int \frac{1}{2\, z\, r'} \frac{du}{\sqrt{u}} \, (r')^2 \, dr'$$

Now integrate the inner integral over u and use Eq. (8) to find the appropriate limits of integration as the polar angle goes from $\theta' = 0$ to $\theta' = \pi$. You should obtain the result

(11)
$$V(z) = \left[\frac{\rho}{2\,\epsilon_o\, z} \int \sqrt{(z^2 + r')^2} - \sqrt{(z^2 - r')^2} \right] \, r' \, dr'$$

To evaluate this integral only take the positive square roots and be sure that you are inside the sphere. Now you need to be very careful here as each of the two integrals in Eq. (11) splits into two separate regimes: one where $0 < r' < z$ and the other where $z < r' < a$, where now z is a point that must be inside the sphere. Use this result to get a familiar result for the electric field $\vec{E}$

(12)
$$\vec{E} = \int \frac{qr}{4\pi\epsilon_o a^3} \, \hat{r}$$

where q is the total charge of the sphere. Note that your answer is identical to the case where the total charge q is concentrated at the center of the sphere which is neat!

Problem 3

Show that the electrostatic potential V of a uniformly charged disk of surface charge density σ and radius a at a distance z on the axis of symmetry is given by

$$V(0,0,z) = \frac{\sigma}{2\epsilon_o} \left[\sqrt{(z^2 - a^2)} - z \right] \quad z > 0$$

$$V(0,0,z) = \frac{\sigma}{2\epsilon_o} \left[\sqrt{(z^2 + a^2)} - z \right] \quad z < 0$$

Carefully discuss its limiting behaviors.

Problem 4

Calculate the electric field $\vec{E}$ *everywhere* at a distance z from the center of a spherical surface of radius a and uniform surface charge density σ. Do not use the electrostatic potential at but use Problem 1 of Problem Set VI as a template for solving this problem.

Problem 5

Find the electrostatic potential V of a uniformly charged solid cylinder at a distance z from the the center. The length of the cylinder is L, its radius is a, and its uniform volume charge density is ρ_V .

$$V(z) = \frac{\rho_V}{4\epsilon_o}\left[-2zL + \left(z+\frac{L}{2}\right)\sqrt{\left(z+\frac{L}{2}\right)^2 + a^2} - \left(z-\frac{L}{2}\right)\sqrt{\left(z-\frac{L}{2}\right)^2 + a^2}\right] +$$

$$\frac{\rho_V}{4\epsilon_o}\, a^2 \ln\left[\frac{(z+\frac{L}{2}) + \sqrt{\left(z+\frac{L}{2}\right)^2 + a^2}}{(z-\frac{L}{2}) + \sqrt{\left(z-\frac{L}{2}\right)^2 + a^2}}\right]$$

Hints:

1. Carefully design your figure and its parameters z, z', and $\vec{r} - \vec{r}\,'$ in this problem and use your figure to relate them. Think about how cylindrical polar coordinates work! Remember that z is always fixed in this problem!

2. You will encounter the following integral to evaluate. First, express it as follows and use a suitable choice of integration by parts to expand it out further

$$\int \sec^3 x \; dx = \int \sec x \; \sec^2 x \; dx$$

3. You will need to use the following relationship to simplify your integral on the right-side

$$1 + \tan^2 x = \sec^2 x$$

4. Given this result you can find a new expression for

$$\int \sec^3 x \; dx$$

5. In this new expression you will encounter an old friend to evaluate

$$\int \sec x \; dx$$

6. You should be able to handle things from here!