(3)
$$\sin\theta'\, d\theta' = \frac{du}{2\, r'\, z}$$

and Eq. (1) becomes

(4)
$$V(z) = \frac{\rho}{2\,\epsilon_o} \int \int \frac{1}{2\, z\, r'} \frac{du}{\sqrt{u}} (r')^2\, dr'$$

Now integrate the inner integral over $u$ and use Eq. (2) to find the appropriate limits of integration as the polar angle goes from $\theta' = 0$ to $\theta' = \pi$ . You should obtain the result

(5)
$$V(z) = \left[\frac{\rho}{2\,\epsilon_o\, z} \int \sqrt{(z^2 + r')^2} - \sqrt{(z^2 - r')^2}\,\right] r'\, dr'$$

To evaluate this integral only take the positive square roots and be sure that you are outside the sphere. You should get a familiar result for the electrostatic potential $V$. Use this result to get a familiar result for the electric field $\vec{E}$

(6)
$$\vec{E} = \int \frac{q}{4\pi\epsilon_o r^2}\, \hat{r}$$

where $q$ is the total charge of the sphere. Note that your answer is identical to the case where the total charge $q$ is concentrated at the center of the sphere which is neat!


Problem 2

In Lecture 9 we calculated the electrostatic potential *inside* a uniformly charged solid sphere of radius $a$ and total charge density $q$. We left out a few steps in our derivation and left them for this problem set.

Let us start with the step where we have done the azimuthal integral

(7)
$$V(z) = \frac{\rho}{2\,\epsilon_o} \int \int \frac{(r')^2\, dr'\, \sin\theta'\, d\theta'}{\sqrt{z^2 + (r')^2 - 2\, z\, r'\, \cos\theta'}}$$

Next we must perform the polar integral over $\theta'$. We first introduce a dummy variable $u$ where

(8)
$$u = z^2 + (r')^2 - 2\, z\, r'\, \cos\theta'$$

When we integrate over the polar angle $\theta'$ we realize from our original figure that $r'$ is fixed . Note also that we treat $z$ as a constant when we do the integral over $\theta'$ since we are evaluating the electrostatic potential $V(z)$ at a fixed value of $z$. Given these observations show that

(3)
$$\sin \theta' d\theta' = \frac{du}{2\, r'\, z}$$

and Eq. (1) becomes

(4)
$$V(z) = \frac{\rho}{2\, \epsilon_o} \int \int \frac{1}{2\, z\, r'} \frac{du}{\sqrt{u}} \, (r')^2 \, dr'$$

Now integrate the inner integral over $u$ and use Eq. (2) to find the appropriate limits of integration as the polar angle goes from $\theta' = 0$ to $\theta' = \pi$. You should obtain the result

(5)
$$V(z) = \left[ \frac{\rho}{2\, \epsilon_o\, z} \int \sqrt{(z^2 + r')^2} - \sqrt{(z^2 - r')^2} \right] r' \, dr'$$

To evaluate this integral only take the positive square roots and be sure that you are outside the sphere. You should get a familiar result for the electrostatic potential $V$. Use this result to get a familiar result for the electric field $\vec{E}$

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$$\vec{E} = \int \frac{q}{4\pi\epsilon_o r^2} \, \hat{r}$$

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Problem 2

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$$V(z) = \frac{\rho}{2\, \epsilon_o} \int \int \frac{(r')^2 \, dr' \, \sin \theta' \, d\theta'}{\sqrt{z^2 + (r')^2 - 2\, z\, r' \, \cos \theta'}}$$

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(8)
$$u = z^2 + (r')^2 - 2\, z\, r' \, \cos \theta'$$

When we integrate over the polar angle $\theta'$ we realize from our original figure that $r'$ is fixed. Note also that we treat $z$ as a constant when we do the integral over $\theta'$ since we are evaluating the electrostatic potential $V(z)$ at a fixed value of $z$. Given these observations show that

(9)
$$\sin\theta'd\theta' = \frac{du}{2\,r'\,z}$$

and **Eq. (7)** becomes

(10)
$$V(z) = \frac{\rho}{2\,\epsilon_o} \int\int \frac{1}{2\,z\,r'}\frac{du}{\sqrt{u}}\,(r')^2\,dr'$$

Now integrate the inner integral over $u$ and use **Eq. (8)** to find the appropriate limits of integration as the polar angle goes from $\theta' = 0$ to $\theta' = \pi$ . You should obtain the result

(11)
$$V(z) = \left[\frac{\rho}{2\,\epsilon_o\,z}\int\sqrt{(z^2+r')^2} - \sqrt{(z^2-r')^2}\,\right]\,r'\,dr'$$

To evaluate this integral only take the positive square roots and be sure that you are inside the sphere. Now you need to be very careful here as each of the two integrals in Eq. (11) splits into two separate regimes: one where $0 < r' < z$ and the other where $z < r' < a$, where now $z$ is a point that must be inside the sphere. Use this result to get a familiar result for the electric field $\vec{E}$

(12)
$$\vec{E} = \int \frac{qr}{4\pi\epsilon_o a^3}\,\hat{r}$$

where $q$ is the total charge of the sphere. Note that your answer is identical to the case where the total charge $q$ is concentrated at the center of the sphere which is neat!

Problem 3

Show that the electrostatic potential $V$ of a uniformly charged disk of surface charge density $\sigma$ and radius $a$ at a distance $z$ on the axis of symmetry is given by

$$V(0,0,z) = \frac{\sigma}{2\epsilon_o}\left[\sqrt{(z^2-a^2)} - z\right]\quad z > 0$$

$$V(0,0,z) = \frac{\sigma}{2\epsilon_o}\left[\sqrt{(z^2+a^2)} - z\right]\quad z < 0$$

Carefully discuss its limiting behaviors.

Problem 4

Calculate the electric field $\vec{E}$ *everywhere* at a distance $z$ from the center of a spherical surface of radius $a$ and uniform surface charge density $\sigma$. Do not use the electrostatic potential at but use Problem 1 of Problem Set VI as a template for solving this problem.

Problem 5

Find the electrostatic potential $V$ of a uniformly charged solid cylinder at a distance $z$ from the the center. The length of the cylinder is $L$, its radius is $a$, and its uniform volume charge density is $\rho_V$.

$$V(z) = \frac{\rho_V}{4\epsilon_o}\left[-2zL + \left(z + \frac{L}{2}\right)\sqrt{\left(z + \frac{L}{2}\right)^2 + a^2} - \left(z - \frac{L}{2}\right)\sqrt{\left(z - \frac{L}{2}\right)^2 + a^2}\right] +$$

$$\frac{\rho_V}{4\epsilon_o}\, a^2 \ln\left[\frac{\left(z + \frac{L}{2}\right) + \sqrt{\left(z + \frac{L}{2}\right)^2 + a^2}}{\left(z - \frac{L}{2}\right) + \sqrt{\left(z - \frac{L}{2}\right)^2 + a^2}}\right]$$

Hints:

1. Carefully design your figure and its parameters $z$, $z'$, and $\vec{r} - \vec{r}'$ in this problem and use your figure to relate them. Think about how cylindrical polar coordinates work! Remember that $z$ is always fixed in this problem!

2. You will encounter the following integral to evaluate. First, express it as follows and use a suitable choice of integration by parts to expand it out further

$$\int \sec^3 x \, dx = \int \sec x \, \sec^2 x \, dx$$

3. You will need to use the following relationship to simplify your integral on the right-side

$$1 + \tan^2 x = \sec^2 x$$

4. Given this result you can find a new expression for

$$\int \sec^3 x \, dx$$

5. In this new expression you will encounter an old friend to evaluate

$$\int \sec x \, dx$$

6. You should be able to handle things from here!

## Problem 1

Starting with (2)

$$U = z^2 + (r')^2 - 2zr'\cos\theta'$$

we realize that $z$ is fixed and $r'$ is also fixed in this expression when we integrate over $\theta$ as seen in our original figure

$$dU = +2zr'\sin\theta' d\theta'$$

Thus

$$\sin\theta' d\theta' = \frac{1}{2zr'} dU$$

which is (3) and (1) becomes

$$V = \frac{\rho}{2\epsilon_0} \iint \frac{1}{2zr'} \frac{dU}{\sqrt{U}} (r')^2 dr'$$

where we have used $U = z^2 + (r')^2 - 2zr'\cos\theta'$

Now integrate over $U$

$$V(z) = \frac{\rho}{4\epsilon_0 \, z \, r'} \iint (r')^2 \, dr' \, U^{-\frac{1}{2}} \, dU$$

or

$$V(z) = \frac{\rho}{4\epsilon_0 \, z} \iint U^{-\frac{1}{2}} \, dU \, r' \, dr'$$

Using $\quad U = z^2 + (r')^2 - 2r'z \cos\theta'$

to obtain the limits of integration

| $\theta'$ | $U$ |
|---|---|
| $0$ | $(z-r')^2$ |
| $\pi$ | $(z+r')^2$ |

$$\int U^{-\frac{1}{2}} \, dU = \frac{\int U^{-\frac{1}{2}+1}}{\frac{1}{2}} = 2\sqrt{U} \Big|_{(z-r')^2}^{(z+r')^2}$$

$$\int U^{-\frac{1}{2}} \, dU = 2\left[\sqrt{(z+r')^2} - \sqrt{(z-r')^2}\right]$$

and $\quad V(z)$ becomes

$$V(z) = \frac{\rho}{4\epsilon_0 z} \, 2 \int \left[ \sqrt{(z+r')^2} - \sqrt{(z-r')^2} \right] r' dr'$$

or

$$V(z) = \frac{\rho}{2\epsilon_0 z} \int \sqrt{(z+r')^2} - \sqrt{(z-r')^2} \, r' dr'$$

We want $z > r'$ in this problem so we must take the positive square roots

$$\sqrt{(z+r')^2} = z + r'$$

$$\sqrt{(z-r')^2} = z - r' \qquad (z > r')$$

and we find

$$V(z) = \frac{\rho}{2\epsilon_0 z} \int \left[ (z+r') - (z-r') \right] r' dr'$$

$$V(z) = \frac{\rho}{2\epsilon_0 z} \int 2(r')^2 dr' = \frac{\rho}{\epsilon_0 z} \frac{(r')^3}{3} \Big|_0^a$$

$$V(z) = \frac{\rho}{\epsilon_0 z} \frac{a^3}{3}$$

Since $\rho = \frac{q \, 3}{4\pi a^3}$

$$V(z) = \frac{3q a^3}{3z\epsilon_0 \, 4\pi a^3} = \frac{q}{4\pi\epsilon_0 z}$$

or

$$V(r) = \frac{q}{4\pi\epsilon_0 r} \quad , \quad r > a$$

which makes sense!

Of course,

$$\vec{E} = -\vec{\nabla} V(\vec{r}) = \frac{+q}{4\pi\epsilon_0 r^2} \hat{r}$$

which also checks out!

Problem 2

Starting with (7)

$$U = z^2 + (r')^2 - 2zr'\cos\theta'$$

we realize that $z$ is fixed and $r'$ is also fixed in this expression when we integrate over $\theta$ as seen in our original figure

$$dU = 2zr'\sin\theta' d\theta'$$

Thus

$$\sin\theta' d\theta' = \frac{1}{2zr'} dU$$

and this is (8) so (7) becomes

$$V = \frac{\rho}{2\epsilon_0} \iint \frac{1}{2zr'} \frac{dU}{\sqrt{U}} (r')^2 dr'$$

where we have used

$$U = z^2 + (r')^2 - 2zr'\cos\theta'$$

From this definition for $u$ we
can do the inner integral over $u$ and evaluate
its limits of integration

| $\theta'$ | $U$ |
|-----------|-----|
| $0$ | $(z - r')^2$ |
| $\pi$ | $(z + r')^2$ |

$$\int u^{-\frac{1}{2}} \, du = \int u^{-\frac{1}{2}} \, du = \left. \frac{u^{-\frac{1}{2}+1}}{\frac{1}{2}} \right|_{(z-r')^2}^{(z+r')^2}$$

$$= 2\left[ \sqrt{(z+r')^2} - \sqrt{(z-r')^2} \right]$$

and $V(z)$ becomes

$$V(z) = \frac{\rho}{(2 z \epsilon_0)} \int \left[ \sqrt{(z+r')^2} - \sqrt{(z-r')^2} \right] r' \, dr'$$

Now there are two regimes over which
we do this integral

Inside the sphere $\quad 0 < r' < z$

and $\quad z < r' < a$

For $\quad 0 < r' < z \quad$, we must have $z > r'$
so

$$\sqrt{(z+r')^2} = +(z+r')$$

$$\sqrt{(z-r')^2} = +(z-r'), \text{ so } z > r'$$

$$V(z) = \frac{\rho}{2 z G_0} \int_0^z \left[ (z+r') - (z-r') \right] r' \, dr'$$

$$V(z) = \frac{\rho}{2 z G_0} \int_0^z 2(r')^2 \, dr' = \frac{\rho}{z G_0} \frac{z^3}{3}$$

For $\quad z < r' < a \quad$ we must have
$\qquad\qquad\qquad a > r' > z$
so

$$\sqrt{(z+r')^2} = +(z+r')$$

$$\sqrt{(z-r')^2} = \sqrt{(r'-z)^2} = +(r'-z),$$
$$\qquad\qquad\qquad r' > z$$

Our integral now becomes

$$V(z) = \frac{\rho}{2z\epsilon_0} \int_z^a \left[ (z+r') - (r'-z) \right] r' \, dr'$$

$$V(z) = \frac{\rho}{2z\epsilon_0} \int_z^a 2z \, r' \, dr'$$

$$V(z) = \frac{\rho}{\epsilon_0} \left. \frac{(r')^2}{2} \right|_z^a$$

$$V(z) = \frac{\rho}{2\epsilon_0} \left[ a^2 - z^2 \right]$$

Combining our two results for $0 < r' < z$ and $z < r' < a$, we find

$$V(z) = \frac{\rho \, z^2}{3\epsilon_0} + \frac{\rho a^2}{2\epsilon_0} - \frac{\rho z^2}{2\epsilon_0}$$

$$V(z) = \frac{2\rho z^2}{6\epsilon_0} - \frac{3\rho z^2}{6\epsilon_0} + \frac{\rho a^2}{2\epsilon_0}$$

$$V(z) = \frac{\rho a^2}{2\epsilon_0} - \frac{\rho z^2}{6\epsilon_0}$$

$$V(z) = \frac{\rho}{2\varepsilon_0} \left[ a^2 - \frac{z^2}{3} \right] , \quad z < a$$

Note where $z = a$

$$V(a) = \frac{\rho}{2\varepsilon_0} \left[ z^2 - \frac{z^2}{3} \right] = \frac{\rho}{2\varepsilon_0} \left[ a^2 - \frac{a^2}{3} \right]$$

$$V(a) = \frac{\rho}{2\varepsilon_0} \left[ \frac{2a^2}{3} \right] = \frac{\rho a^2}{3\varepsilon_0}$$

and for $\rho = \frac{q \cdot 3}{4\pi a^3}$

$$V(a) = \frac{3q}{4\pi a^3} \frac{a^2}{3\varepsilon_0} = \frac{q}{4\pi \varepsilon_0 a}$$

which makes
sense!

Finally if

$$V(z) = \frac{\rho}{2\epsilon_0} \left[ a^2 - \frac{z^2}{3} \right]$$

we can express our result entirely in terms of q

$$\rho = \frac{q}{4\pi a^3}^3$$

$$V(z) = \frac{q^3}{4\pi a^3} \frac{1}{2\epsilon_0} \left[ a^2 - \frac{z^2}{3} \right]$$

$$V(z) = \frac{3q}{8\pi a^3 \epsilon_0} \left[ a^2 - \frac{z^2}{3} \right]$$

$$\boxed{V(z) = \frac{q}{8\pi \epsilon_0 a} \left[ 3 - \frac{z^2}{a^2} \right]} \quad \begin{array}{l} \text{For} \\ z < a \end{array}$$

What is $\vec{E}$?

$$\frac{\partial V}{\partial z} = -\frac{2z}{a^2} \frac{q}{8\pi \epsilon_0 a} = \frac{-qz}{4\pi \epsilon_0 a^3}$$

$$\vec{E} = + \frac{qr}{4\pi \epsilon_0 a^3} \hat{r} \quad \text{which makes}$$
$$\text{perfect sense!}$$

Problem # 3

Starting with our coordinate system
in cylindrical polar coordinates
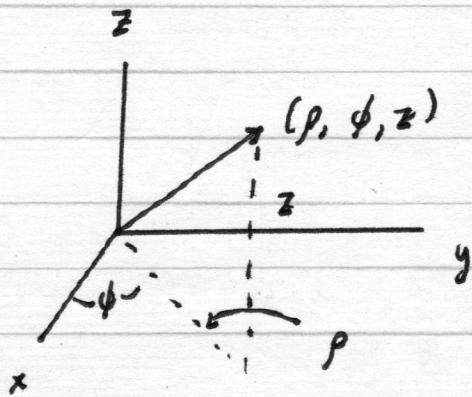


We can use them for our problem
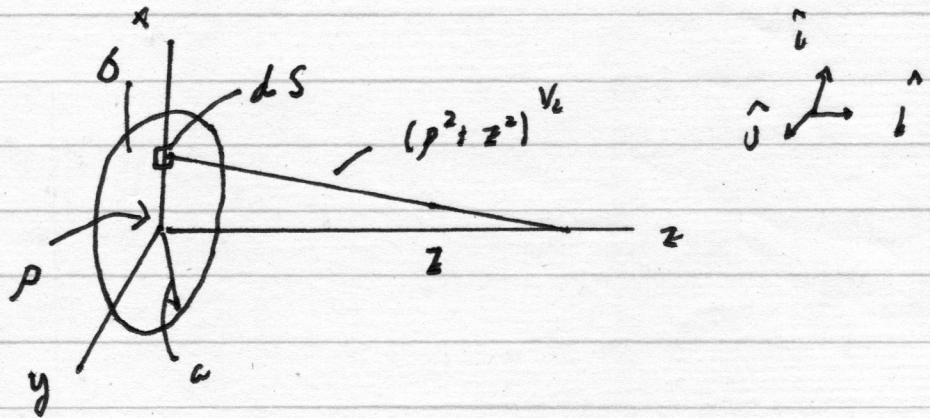


and proceed

$$V(0,0,z) = \int \frac{dq}{4\pi \epsilon_0 |\vec{r} - \vec{r}'|} \qquad \int \frac{\sigma \, dS}{4\pi \epsilon_0 |\vec{r} - \vec{r}'|} \quad (4-1)$$



$$V(0,0,z) = \frac{\sigma}{4\pi \epsilon_0} \int \frac{\rho \, d\rho \, d\phi}{|\vec{r} - \vec{r}'|} \quad (4-2)$$

Origin of
disk
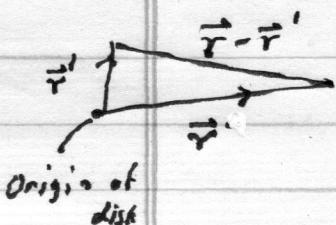
From our figure

$$|\vec{r}-\vec{r}'| = [\rho^2 + z^2]^{1/2} \qquad (4-3)$$

so $(4-2)$ becomes

$$V(z) = \frac{\sigma}{4\pi \epsilon_0} \int_S \frac{\rho \, d\rho \, d\phi}{[\rho^2 + z^2]^{1/2}} \qquad (4-4)$$

let us integrate out the azimuthal angle

$$V(z) = \frac{\sigma}{2\epsilon_0} \int \frac{\rho \, d\rho}{[\rho^2 + z^2]^{1/2}} \qquad (4-5)$$

Now since $z$ is fixed we are in business

$$U = \rho^2 + z^2 \qquad (4-6)$$

$$dU = 2\rho \, d\rho \qquad (4-7)$$

and

| $\rho$ | $U$ |
|---|---|
| $0$ | $z^2$ |
| $a$ | $z^2 + a^2$ |
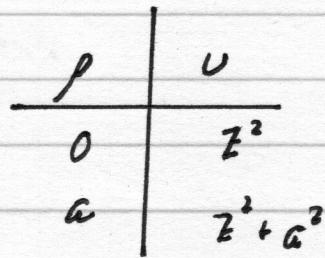
so we can take into account our limits
of integration

$$(4-8) \qquad V(z) = \frac{\sigma}{2\epsilon_0} \int_{z^2}^{z^2+a^2} \frac{1}{2} \frac{du}{\sqrt{u}} = \frac{\sigma}{4\epsilon_0} \left[ \frac{u^{-\frac{1}{2}+1}}{\frac{1}{2}} \right]_{z^2}^{z^2+a^2}$$

$$(4-9) \qquad V(z) = \frac{\sigma}{2\epsilon_0} \left[ (z^2+a^2)^{\frac{1}{2}} - (z^2)^{\frac{1}{2}} \right]$$

Note that in our figure $z > 0$, so

$$(4-10) \qquad V(z) = \frac{\sigma}{2\epsilon_0} \left[ (z^2+a^2)^{\frac{1}{2}} - z \right], \quad z > 0$$

By symmetry the electrostatic potential $V(z)$
must have the same value for $z < 0$.
If we take the negative square root
of $(z^2)^{\frac{1}{2}}$ (as opposed to the positive square root
as we did in $(4-9)$, we have

$$(4-11) \qquad (z^2)^{\frac{1}{2}} = -z$$

which must give the same physical
result for

$$(4-12) \quad z < 0$$

(4-13)

$$V(z) = \frac{\sigma}{2\epsilon_0} \left[ (z^2 + a^2)^{1/2} + z \right], \quad z < 0$$

Thus

(4-14) $\quad V(z) = \frac{\sigma}{2\epsilon_0} \left[ (z^2 + a^2)^{1/2} - z \right], \quad z > 0$

(4-15) $\quad V(z) = \frac{\sigma}{2\epsilon_0} \left[ (z^2 + a^2)^{1/2} + z \right], \quad z < 0$

Finally, let us explore the limiting cases.

① Note for $z = 0$ $\quad V = \frac{\sigma a}{2\epsilon_0} = \frac{q}{\pi a^2} \frac{a}{2\epsilon_0} = \frac{q}{2\pi\epsilon_0 a}$

② For $z \gg a$

$$\left[ (z^2 + a^2)^{1/2} - z \right] = \left[ z \left( 1 + \left(\frac{a}{z}\right)^2 \right)^{1/2} - z \right]$$

$$= z \left[ \left( 1 + \left(\frac{a}{z}\right)^2 \right)^{1/2} - 1 \right]$$

You can expand

$$(1 + x)^{1/2}$$

as a binomial series

$$\left(1 + \frac{1}{2}x + \frac{1}{2!} \frac{1}{2}\left(\frac{1}{2} - 1\right) x^2 + \cdots\right)$$

which for small values of x only becomes important for lower order terms

$$(1 + x)^{1/2} \simeq 1 + \frac{1}{2}x + \cdots$$

Thus for $a < z$

$$\left[(z^2 + a^2)^{1/2} - z\right] = z\left[\left(1 + \left(\frac{a}{z}\right)^2\right)^{1/2} - 1\right]$$

$$\simeq z\left[1 + \frac{1}{2}\left(\frac{a}{z}\right)^2 + \cdots - 1\right]$$

$$\simeq \frac{1}{2}\left(\frac{a}{z}\right)^2$$

and

$$\left[(z^2 + a^2)^{1/2} - z\right] \simeq z\left(\frac{1}{2}\frac{a^2}{z^2}\right) \simeq \frac{a^2}{2z}$$

and

$$V(z) = \frac{\sigma}{2\epsilon_0} \cdot \frac{a^2}{2z} \approx \frac{\sigma}{4\epsilon_0} \frac{a^2}{z}$$

or

$$V(z) = \frac{\sigma}{4\epsilon_0} \frac{a^2}{z} = \frac{q}{4\pi a^2 \epsilon_0} \frac{a^2}{z}$$

$$V(z) \longrightarrow \frac{q}{4\pi\epsilon_0 z} \qquad \text{as} \quad z \gg a$$

which makes perfect sense.

We should note that the binomial series is just an example of a Taylor series. If you need a review of

① Maclaurin series
② Taylor series
③ Binomial expansion
④ Binomial series

_I_ would strongly recommend Khan Academy for an excellent discussion of these topics!

Problem 4

Find $\vec{E}$ everywhere along $z$ !
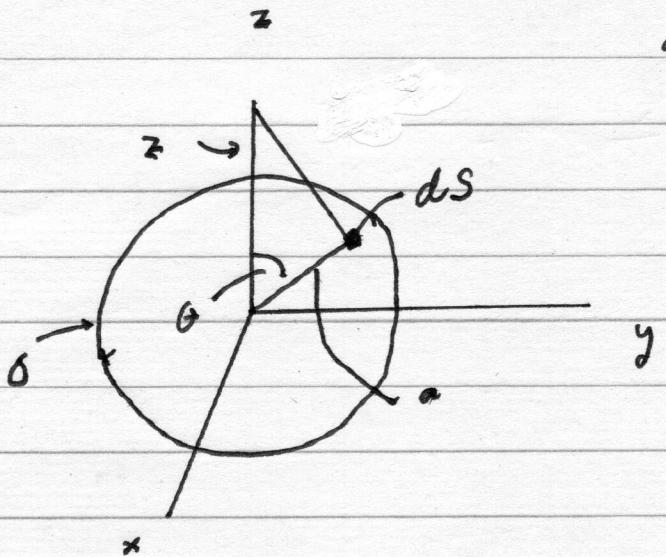


Using Problem 1 of Problem Set VI as a template symmetry says there is only one component that contributes to $\vec{E}$

$$(4\cdot1) \qquad E_{\bar{z}} = \int \frac{dq \, \cos\alpha}{4\pi\epsilon_0 \, |\vec{r}-\vec{r}'|^2}$$

where

$$|\vec{r}-\vec{r}'| = s$$
$$s^2 = |\vec{r}-\vec{r}'|^2 = \bar{z}^2 + a^2 - 2a\bar{z}\cos\theta$$
$$(4-2)$$
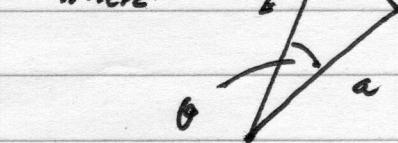
and

$$(4-3) \qquad a^2 = \bar{z}^2 + s^2 - 2\bar{z}s\cos\alpha$$

$$s = |\vec{r}-\vec{r}'|$$

(N.B. $s \neq S$
$$\underset{\text{area}}{S} \qquad |\vec{r}-\vec{r}'|$$

It follows from spherical polar coordinates that

$$(4\text{-}4) \qquad dq = \sigma a^2 \sin\theta \, d\theta \, d\phi = \sigma dS$$

and

$$(4\text{-}5) \qquad E_z = \int \frac{\sigma a^2}{4\pi\epsilon_0} \frac{\sin\theta \, d\theta \, d\phi \cos\alpha}{S^2}$$

To simplify the integrand a little we observe from (4-3) that

$$(4\text{-}6) \qquad \cos\alpha = \frac{z^2 + S^2 - a^2}{2zS}$$

and upon doing the azimuthal integral in (4-5) we find

$$(4\text{-}7) \qquad E_z = \frac{\sigma a^2}{4\epsilon_0 z} \iint \frac{\sin\theta \, d\theta \, [z^2 + S^2 - a^2]}{S^3} dS$$

where $z$ is fixed and we only have two variables of integration $\theta, S$

If we perform the polar integration over $\theta$, then we obtain from (4-2)

$$(4-8) \qquad -2az \cos \theta = s^2 - z^2 - a^2$$

or

$$(4-9) \qquad +\sin \theta \, d\theta = d\left[\frac{s^2 - z^2 - a^2}{2az}\right] = \frac{2s \, ds}{2az},$$

or

$$(4-10) \qquad \sin \theta \, d\theta = \frac{s \, ds}{az}$$

so (4-7) simplifies to

$$(4-11) \qquad E_z = \frac{\sigma a^2}{4 \epsilon_0 z} \int \frac{ds}{az s^2} \left[ z^2 + s^2 - a^2 \right]$$

or

$$(4-12) \qquad E_z = \frac{\sigma a}{4 \epsilon_0 z^2} \int ds \left[ \frac{z^2 - a^2}{s^2} + 1 \right]$$

Now we need find the limits of integration over $s$

From (4-2) we have

(4-13) $\qquad s^2 = z^2 + a^2 - 2az \cos \theta$

and from our figure we see that we are integrating $\theta$ from $0$ to $\pi$ so that

For $\theta = 0$ $\qquad S^2 = (z-a)^2$ $\qquad$ (4-14)

For $\theta = \pi$ $\qquad S^2 = (z+a)^2$ $\qquad$ (4-15)

We first will evaluate (4-12) for $z > a$ so we take the positive square roots to find

(4-16) $\qquad s = (z-a)$ for $\theta = 0$

(4-17) $\qquad s = (z+a)$ for $\theta = \pi$

and (4-12) becomes

(4-18) $\qquad E_z = \dfrac{\delta a}{4G_0 z^2} \displaystyle\int_{(z-a)}^{(z+a)} \left[ \dfrac{z^2 - a^2}{s^2} + 1 \right] ds$

This integral is now straightforward to do

$$(4\text{-}19) \qquad E_z = \frac{\sigma a}{4\epsilon_0 z^2}\left[ -\frac{(z^2-a^2)}{S} + S \right]_{(z-a)}^{(z+a)}$$

or

$$(4\text{-}20) \qquad E_z = \frac{\sigma a}{4\epsilon_0 z^2}\left[ (z+a) - \frac{(z^2-a^2)}{(z+a)} \right.$$
$$\left. - (z-a) + \frac{(z^2-a^2)}{(z-a)} \right]$$

or

$$(4\text{-}21) \qquad E_z = \frac{\sigma a}{4\epsilon_0 z^2}\left[ z+a -(z-a) \right.$$
$$\left. -(z-a) + z+a \right]$$

or

$$(4\text{-}22) \qquad E_z = \frac{\sigma a}{4\epsilon_0 z^2}\left[ \cancel{z} - \cancel{z} + a + a - \cancel{z} + a \right.$$
$$\left. \cancel{z} + a \right]$$

or

$$(4\text{-}23) \qquad E_z = \frac{\sigma a^2}{\epsilon_0 z^2}$$

Finally since

$$\delta = \frac{q}{4\pi a^2}$$

(4-24) $\qquad E_z = \frac{q}{4\pi \epsilon_0 z^2}$

or

$$\boxed{(4.25) \quad \vec{E} = \frac{q}{4\pi \epsilon_0 r^2} \hat{r} \quad , \text{ for } r > a}$$

which makes sense!

Now we return to (4.14) to study the case where $z < a$. In order to have a positive square root we must realize

(4-29) $\qquad$ For $\theta = 0 \qquad s^2 = (z-a)^2$

or

$$s = \sqrt{(z-a)^2} = \sqrt{(a-z)^2} =$$

$$a - z$$

(4-30) $\qquad$ For $\theta = \pi \qquad s^2 = (z+a)^2$

$$s = \sqrt{(z+a)^2} = z + a$$

and (4-19) becomes

$$(4\text{-}31) \qquad E_z = \frac{\sigma a}{4\epsilon_0 z^2} \left[ S - \frac{(z^2 - a^2)}{S} \right]_{(a-z)}^{(z+a)}$$

Evaluating this yields

$$(4\text{-}32) \qquad E_z = \frac{\sigma a}{4\epsilon_0 z^2} \left[ (z+a) - \frac{(z^2 - a^2)}{(z+a)} \right.$$

$$\left. - (a-z) + \frac{(z^2 - a^2)}{(a-z)} \right]$$

or

$$(4\text{-}33) \qquad E_z = \frac{\sigma a}{4\epsilon_0 z^2} \left[ (z+a) - (z-a) \right.$$

$$\left. - (a-z) - (a+z) \right]$$

$$(4\text{-}34) \qquad E_z = \frac{\sigma a}{4\epsilon_0 z^2} \left[ \cancel{z} - \cancel{z} + \cancel{a} + \cancel{a} - \cancel{a} + \cancel{z} \right.$$

$$\left. - \cancel{a} - \cancel{z} \right]$$

$$(4\text{-}35) \qquad \vec{E} = 0 \;,\quad z < a \quad \left( \begin{array}{l}\text{which makes sense} \\ \text{from Gauss's law}\end{array} \right)$$

## Problem 5

Find the electrostatic potential $V$ on the axis of a uniformly charged solid cylinder, a distance $z$ from the center. The length of this cylinder is $L$, its radius $a$, and its charge density $\rho_V$. Use your assumption that $z > L/2$ in this problem, carefully!
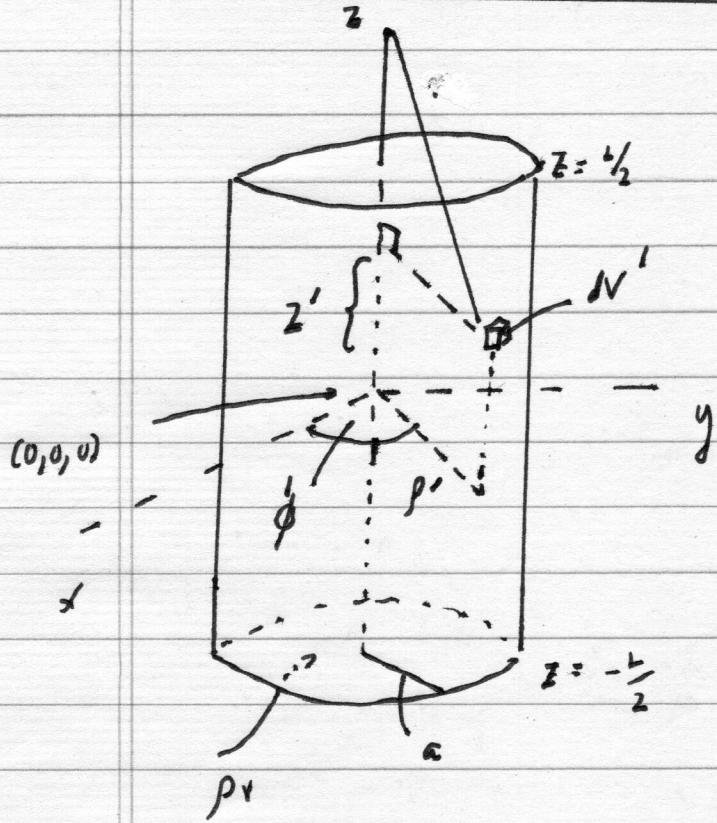


Let us exploit cylindrical polar coordinates to study the problem and draw the solid cylinder carefully.

Some observations:

① $\rho_V$ = volume charge density

② $\rho$ = coordinate in cylindrical polar coordinates

③ $\rho \neq \rho_V$

④ The cylinder is centered at $(0,0,0)$.

⑤ The distance $z$ from the origin to a point outside the cylinder is a constant.

⑥ The cylinder has a height from $z = -\frac{L}{2}$ to $z = +\frac{L}{2}$ or $L$.

⑦ The radius of the cylinder is $a$ (constant)

Now let us be careful!

⑧ The differential volume element of $dV'$ is

$$dV' = \rho' \, d\rho' \, d\phi' \, dz'$$

where we use primed variables to refer to the charge distribution. Note $z'$ is not constant and it tells you where $dV'$ is located with respect to the origin!

⑨ How are $\rho'$, $z'$, and $z$ related?

Look carefully at our figure!
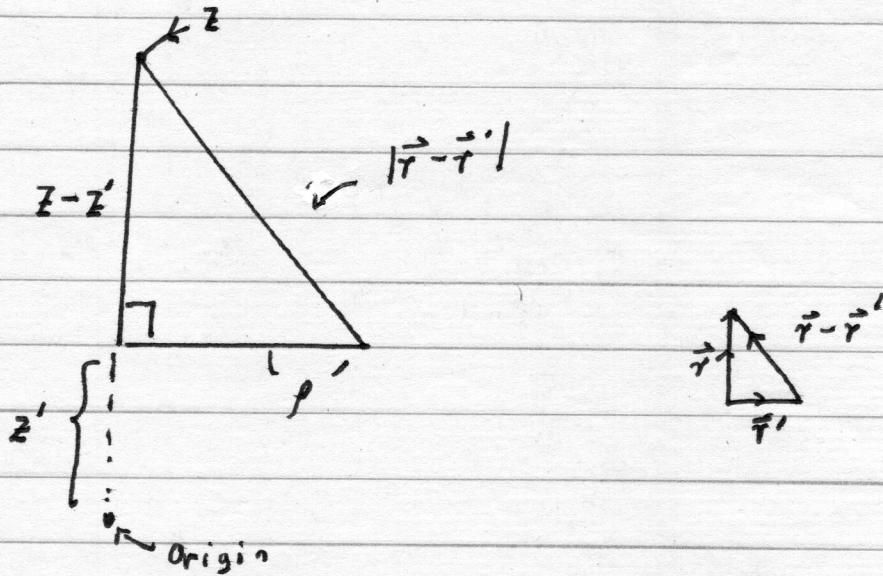


Note that $z$ is greater than $b/2$ or outside
the cylinder. We will return to this point later.
From our above figure

$$|\vec{r} - \vec{r}'| = \left[ (\rho')^2 + (z - z')^2 \right]^{1/2}$$ , $z$ is fixed
$z'$ is a variable

This makes sense as for $z' = z$

$$|\vec{r} - \vec{r}'| = \rho'$$

and for $z' = 0$

$$|\vec{r} - \vec{r}'| = \left[ (\rho')^2 + z^2 \right]^{1/2}$$

Clearly $z' + (z - z') = z$

Make sure you see this
from figure!!

⑲ Finally we can modify ⑥. The variable $z'$ must give us information about the height of the cylinder so it goes from $x' = -\frac{L}{2}$ to $x' = \frac{L}{2}$

Now we are ready to solve our problem

(1) $\quad V = \int \frac{dq}{4\pi \epsilon_0 |\vec{r} - \vec{r}'|} = \frac{\rho_v}{4\pi \epsilon_0} \int_v \frac{\rho' d\rho' d\phi' dz'}{\left[ (\rho')^2 + (z - z')^2 \right]^{\frac{1}{2}}}$

Doing the azimuthal angle first gives

(2) $\quad V(z) = \frac{\rho_v}{2\epsilon_0} \int_s \frac{\rho' d\rho' dz'}{\left[ (\rho')^2 + (z - z')^2 \right]^{\frac{1}{2}}}$

or

(3) $\quad V(z) = \frac{\rho_v}{2\epsilon_0} \int \int \frac{\rho' d\rho' dz'}{\left[ (\rho')^2 + (z - z')^2 \right]^{\frac{1}{2}}}$

Let us evaluate the inner integral over $\rho'$ first

(4) $\qquad V(z) \sim \int \dfrac{\rho' d\rho'}{[(\rho')^2 + (z-z')^2]^{1/2}}$

where $z$ is of course a constant and we hold $z'$ fixed while performing this integral

Let us define

(5) $\qquad U = (\rho')^2 + (z-z')^2$

where

(6) $\qquad dU = 2(\rho') d\rho'$

Our integral (inner) becomes

(7) $\dfrac{1}{2} \int \dfrac{dU}{U^{1/2}} = \dfrac{1}{2} \dfrac{U^{-1/2+1}}{1-\frac{1}{2}} \Big| = \sqrt{U} \Big|$

Now we need to evaluate the limits of integration for $U$

$$\begin{array}{c|c} \rho' & U \\ \hline 0 & (z-z')^2 \\ a & a^2 + (z-z')^2 \end{array}$$
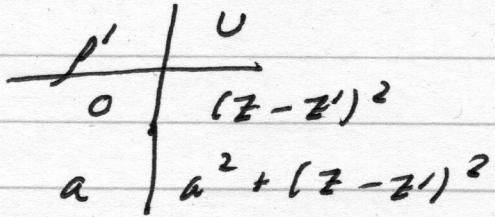
so we can evaluate (7) and place it back into (3) to yield

(8) $\quad V(z) = \dfrac{\rho v}{2\epsilon_0} \displaystyle\int \left[ \sqrt{(z-z')^2 + a^2} - \sqrt{(z-z')^2} \right] dz'$

or

(9) $\quad V(z) = \dfrac{\rho v}{2\epsilon_0} \displaystyle\int_{-\frac{L}{2}}^{\frac{L}{2}} \left[ \sqrt{(z-z')^2 + a^2} - \sqrt{(z-z')^2} \right] dz'$

For the moment let us ignore the limits of integration and tackle the first integral

(10) $\quad \displaystyle\int \sqrt{(z-z')^2 + a^2} \; dz'$

Let $\quad U = z - z' \quad$ (11) $\quad dU = -dz' \quad$ (12)

$\qquad\qquad$ (N.B. $z > z'$, always )

and we get

(13) $\quad -\displaystyle\int \sqrt{U^2 + a^2} \; dU$

Now we need to evaluate this integral

(14)
$$- \int \sqrt{v^2 + a^2} \, dv$$

where $a$ is a constant. Let us try trigonometric substitution

$$- \int a \sqrt{1 + \frac{v^2}{a^2}} \, dv$$

$$\tan \theta = \frac{v}{a} \qquad v = a \tan \theta$$
$$dv = a \sec^2 \theta \, d\theta$$

$$[1 + \tan^2 \theta]^{1/2} = \sec \theta$$

$$- \int a \sqrt{1 + \frac{v^2}{a^2}} \, dv = = - \int a \sec \theta \, a \sec^2 \theta \, d\theta$$

$$= - a^2 \int \sec^3 \theta \, d\theta$$

This integral is a bit challenging but we can use a few clever observations

$$\int sec^3\theta \, d\theta = \int sec\theta \, sec^2\theta \, d\theta$$

Now integrate by parts

Let $\quad U = sec\theta$

$\quad dv = sec^2\theta \, d\theta$

$\quad v = tan\theta$

$\quad du = \dfrac{-(-sin\theta)d\theta}{cos^2\theta} = (tan\theta \, sec\theta \, d\theta)$

so

$$\int sec^3\theta \, d\theta = sec\theta \, tan\theta - \int tan^2\theta \, sec\theta \, d\theta.$$

Now using the fact that

$$1 + tan^2\theta = sec^2\theta \quad our \; last$$

integral becomes

$$\int sec^3\theta \, d\theta = sec\theta \, tan\theta - \int sec^3\theta \, d\theta$$

$$+ \int sec\theta \, d\theta$$

or

$$2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \int \sec \theta \, d\theta$$

Finally we have seen this last integral before in this course

$$2 \int \sec^3 \theta \, d\theta = \sec \theta \tan \theta + \ln \left[ \tan \theta + \sec \theta \right]$$

so

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} \left[ \sec \theta \tan \theta + \ln \left[ \tan \theta + \sec \theta \right] \right]$$

We can get rid of the trigonometric functions as follows

$$\tan \theta = \frac{U}{a}$$
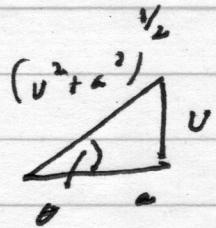


$$\sin \theta = \frac{U}{(U^2 + a^2)^{1/2}}$$

$$\cos \theta = \frac{a}{(U^2 + a^2)^{1/2}}$$

so that we have

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} \left[ \frac{U}{a} \quad \frac{(U^2+a^2)^{1/2}}{a} \right. +$$

$$\ln \left[ \frac{U}{a} + \frac{(U^2+a^2)^{1/2}}{a} \right] \right]$$

and using (11)

(15)
$$\int \sec^3 \theta \, d\theta = \frac{1}{2} \left[ \left( \frac{z-z'}{a} \right) \frac{[(z-z')^2+a^2]^{1/2}}{a} \right.$$

$$+ \ln \left[ \frac{(z-z')}{a} + \frac{[(z-z')^2+a^2]^{1/2}}{a} \right] \right]$$

Thus our first integral (11) becomes

(16)
$$\int \sqrt{(z-z')^2+a^2} \, dz' = -a^2 \int \sec^3 \theta \, d\theta$$

$$= -\frac{a^2}{2} \left[ \left( \frac{z-z'}{a} \right) \left( \frac{[(z-z')^2+a^2]^{1/2}}{a} \right) \right.$$

$$+ \ln \left[ \frac{(z-z')}{a} + \frac{[(z-z')^2+a^2]^{1/2}}{a} \right] \right]$$

Now we are in a position to evaluate the limits of integration of $z'$ from $-L/2$ to $+L/2$

(17)
$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{(z-z')^2 + a^2}\, dz'$$

$$= -\frac{a^2}{2}\left[ \left(\frac{z-L/2}{a}\right)\left(\frac{[(z-L/2)^2 + a^2]^{1/2}}{a}\right) \right.$$

$$+ \ln\left[ \frac{z-L/2}{a} + \frac{[(z-L/2)^2 + a^2]^{1/2}}{a} \right]\Bigg]$$

$$+ \frac{a^2}{2}\left[ \left(\frac{z+L/2}{a}\right)\left(\frac{[(z+L/2)^2 + a^2]^{1/2}}{a}\right) \right.$$

$$+ \ln\left[ \frac{z+L/2}{a} + \frac{[(z+L/2)^2 + a^2]^{1/2}}{a} \right]\Bigg]$$

or

(18)
$$\int_{-L/2}^{L/2} \sqrt{(z-z')^2 + a^2}\ dz' =$$

$$\frac{a^2}{2}\left[\left(\frac{z+L/2}{a}\right)\left(\frac{[(z+L/2)^2 + a^2]^{1/2}}{a}\right)\right.$$

$$-\ \left(\frac{z-L/2}{a}\right)\left(\frac{[(z-L/2)^2 + a^2]^{1/2}}{a}\right)$$

$$\left. +\ \ln\left[\frac{(z+L/2) + [(z+L/2)^2 + a^2]^{1/2}}{(z-L/2) + [(z-L/2)^2 + a^2]^{1/2}}\right]\right]$$

Thus the contribution of the first integral to ① is

(19)
$$\frac{\rho_r}{G_0}\frac{a^2}{2}\left[\left(\frac{z+L/2}{a}\right)\left(\frac{[(z+L/2)^2 + a^2]^{1/2}}{a}\right)\right.$$

$$-\ \frac{(z-L/2)}{a}\left(\frac{[(z-L/2)^2 + a^2]^{1/2}}{a}\right)$$

$$\left. +\ \ln\left[\frac{(z+L/2) + [(z+L/2)^2 + a^2]^{1/2}}{(z-L/2) + [(z-L/2)^2 + a^2]^{1/2}}\right]\right]$$

The second integral in ⑨ can easily be evaluated as follows

⑳ $$-\frac{\rho_V}{2\epsilon_0} \int_{-L/2}^{L/2} \sqrt{(z-z')^2} \, dz'$$

$U = z - z' \implies dz' = -dU$ since

$$z \text{ is a constant}$$

$$\sqrt{(z-z')} = \sqrt{U^2} = \pm U$$

but we take the positive square root so $z > z'$

㉑ $$+\frac{\rho_V}{2\epsilon_0} \int_{-L/2}^{L/2} \sqrt{(z-z')^2} \, dz'$$

㉒ $$\frac{\rho_V}{2\epsilon_0} \int \sqrt{U^2} \, dU$$

| $z'$ | $U$ |
|------|-----|
| $-L/2$ | $z + L/2$ |
| $+L/2$ | $z - L/2$ |

(23) $\quad 2\left(\dfrac{\rho v}{G_0}\right)\dfrac{u^2}{2}\ \Bigg|_{z+\frac{L}{2}}^{z-\frac{L}{2}}$

(24) $\quad \dfrac{\rho v}{2G_0(z)}\left[\left(z-\tfrac{L}{2}\right)^2 - \left(z+\tfrac{L}{2}\right)^2\right]$

(25) $\quad \dfrac{\rho v}{2G_0}\left[z^2 + \dfrac{L^2}{4} - zL - \left(z^2 + \dfrac{L^2}{4} + zL\right)\right]$

(26) $\quad \dfrac{\rho v}{2G_0}\left[-2zL\right]$

If we combine (26) + (19) we
obtain the final result

$$\text{(27)} \quad V(z) = \frac{\rho_v}{4G_i} \left[ (z + \tfrac{L}{2}) \left[ (z + \tfrac{L}{2})^2 + a^2 \right]^{1/2} \right.$$

$$- (z - \tfrac{L}{2}) \left[ (z - \tfrac{L}{2})^2 + a^2 \right]^{1/2} - 2zL$$

$$+ a^2 \ln \left[ \frac{(z + \tfrac{L}{2}) + \left[ (z + \tfrac{L}{2})^2 + a^2 \right]^{1/2}}{(z - \tfrac{L}{2}) + \left[ (z - \tfrac{L}{2})^2 + a^2 \right]^{1/2}} \right] \right]$$