USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET IV
(due Monday, July 20, 2020)

Problem 0

Please note the typographical error in Problem 14 of Problem Set I. The correct expression is listed below. The solution provided is correct despite the typographical error where the problem is initially stated.

$$\hat{\imath} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

Problem 1

Find the expression for the gradient of a scalar function $f(x,y,z)$ in cylindrical polar coordinates. You should start with the gradient operator in Cartesian coordinates

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

and operate on the scalar function $f(x,y,z)$ to obtain the following partial derivatives

$$\frac{\partial f}{\partial x} = \cos\phi\frac{\partial f}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial f}{\partial \phi}$$

$$\frac{\partial f}{\partial y} = \sin\phi\frac{\partial f}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial f}{\partial \phi}$$

You will need to recall the following useful information

$$\rho = \sqrt{x^2 + y^2}$$

$$\tan\phi = \frac{y}{x}$$

and you will need to remember from elementary calculus how to do implicit differentiation of partial derivatives and the chain rule, of course. For example, consider the function

$$y = x\tan\phi$$

where

$$\phi = \phi(x, y)$$

$$\frac{\partial y}{\partial y} = 1 = x\sec^2\phi\,\frac{\partial\phi}{\partial y}$$

or

$$\frac{\partial\phi}{\partial y} = \frac{1}{x\,\sec^2\phi}$$

Here is your final desired result

$$\nabla f = \hat{\rho}\frac{\partial f}{\partial\rho} + \hat{\phi}\,\frac{1}{\rho}\,\frac{\partial f}{\partial\phi} + \hat{k}\,\frac{\partial f}{\partial z}$$

Problem 2

Find the expression for the gradient of a scalar function $f(x, y, z)$ in spherical polar coordinates. Here is your desired result

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

You should start with the gradient operator in Cartesian coordinates and proceed in the same fashion as in Problem 1. Of course, you will have to do more work! Here are some useful helpful hints

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\tan\phi = \frac{y}{x}$$

$$\frac{\partial f}{\partial x} = \sin\theta\,\cos\phi\,\frac{\partial f}{\partial r} + \frac{\cos\theta\,\cos\phi}{r}\frac{\partial f}{\partial \theta} - \frac{\sin\phi}{\rho}\frac{\partial f}{\partial \phi}$$

$$\frac{\partial f}{\partial y} = \sin\theta\,\sin\phi\,\frac{\partial f}{\partial r} + \frac{\cos\theta\,\sin\phi}{r}\frac{\partial f}{\partial \theta} + \frac{\cos\phi}{\rho}\frac{\partial f}{\partial \phi}$$

$$\frac{\partial f}{\partial z} = \cos\theta\,\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

Problem 3

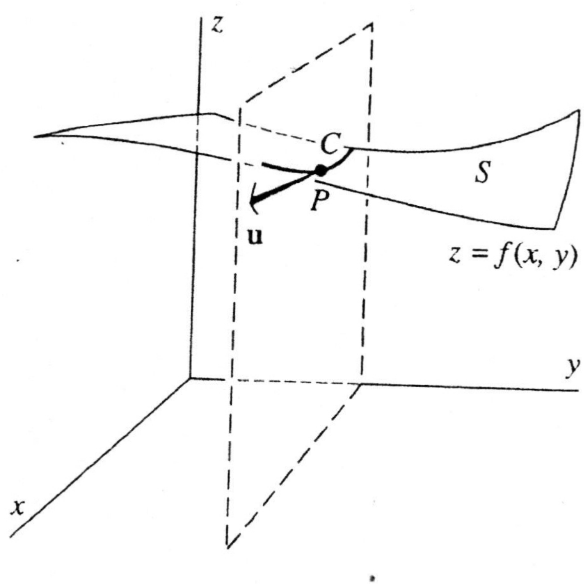
The ideas of the directional derivative and the gradient of a function can be easily understood using simple pictures. First, please start using *Wolfram Alpha* which has a free version that can be accessed using Google. Start playing around with the plot command to visualize the surface of a function of two variables $z = f(x, y)$ for the following functions

$$f(x, y) = \sin x \cos y$$

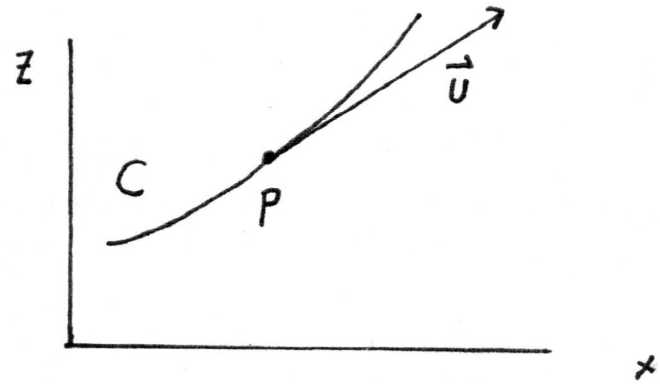
$$f(x, y) = x^2y^3, x = -1..1, y = 0..3$$

$$f(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

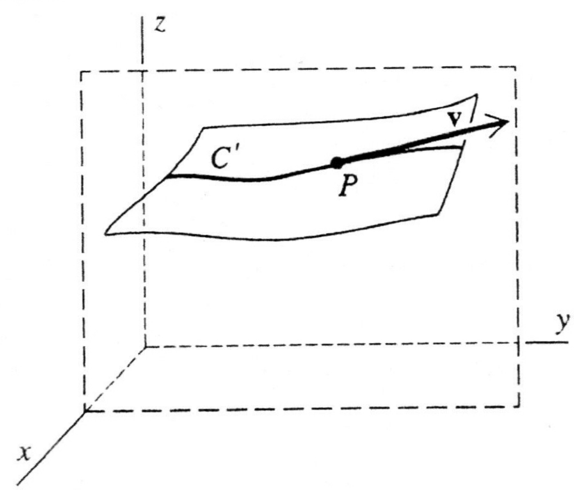
If you choose any point (x, y) for any one of these smooth, well-behaved functions you get a point z which is unique. The question is what does it mean to ask what the slope is at that particular point z? Let us consider the surface S in the figure below



In this figure S is some arbitrary surface defined by the function $z = f(x, y)$. If you pick a point P on the surface S and construct a plane through the point P on S and parallel to the xz-plane, that plane intersects the surface S in a curve C. By looking at the figure below it is clear that we can define a vector $\vec{u}$ which is tangent to the curve C at our point P. This is just the slope of the curve C at the point P which is our normal definition of the derivative of a function of a single variable.



The problem is that this process is not unique as there are many other possible planes you could construct and thus find a different derivative. In fact there are an infinite number of them! For example, see the figure below for another choice of the plane through the point P to obtain a different curve C' and thus a different vector $\vec{v}$



The derivative at a point P on a surface S depends our which direction you want to pick. Thus this is why it is called a *directional derivative.* Now pictures are nice but we wish to know how to actually calculate a directional derivative for a particular surface S defined by the function $z = f(x, y)$.

So now let us formalize this a bit. Note that this discussion is slightly different from what we did in Lecture 4. If we are given a function $f(x, y, z)$ we can compute the total change in df by using partial derivatives

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

As discussed in Lecture 4, this is simply the dot product of two vectors

$$df = \nabla f \cdot d\vec{r}$$

where ∇ is the gradient operator

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

and $d\vec{r}$ is the displacement vector. Since $d\vec{r} = \hat{r}\, dr$ we can write

$$\frac{df}{dr} = \nabla f \cdot \hat{r}$$

and this is our formal definition of the directional derivative of $f(x, y, z)$. For example, if we choose the direction

$$\hat{r} = \hat{\imath}$$

$$\frac{df}{dr} = \nabla f \cdot \hat{\imath}$$

and

$$\frac{df}{dr} = \frac{\partial f}{\partial x}$$

which makes perfect sense. Note that our directional derivative can be defined for any direction not just the direction of a unit vector. That is what makes it so powerful a concept! Now let us tackle the physical meaning of the gradient term in the equation

$$\frac{df}{dr} = \nabla f \cdot \hat{r}$$

The directional derivative at any given point on the surface defined by $f(x, y, z)$ has many values (actually an infinite number) depending on the angle θ. When you evaluate the dot product on the right hand side of the equation you obtain

$$\frac{df}{dr} = \nabla f \cdot \hat{r} = \| \nabla f \| \cos\theta$$

It is easy to see that the directional derivative that has the greatest or maximum increase occurs when θ is 0 or when ∇f and $\hat{r}$ both point in the same direction. Thus the gradient of a function $\nabla f(x,y,z)$ is a vector that points in the direction of the maximum increase of that function. It is a very special directional derivative. The norm of the gradient of a function $\nabla f(x,y,z)$ is the rate of increase or slope along this maximal direction.

There is more useful information that we can obtain from the gradient of a function $\nabla f(x,y,z)$. If $\nabla f(x,y,z)$ were to vanish at a point (x,y,z), then $df = 0$ for small displacements about the point (x,y,z). This is therefore a *stationary point* of the function $f(x,y,z)$ and it could be a summit (maximum), a valley (a minimum), or a pass (saddle point). This is similar to the case of a function of a single variable, where a vanishing derivative implies either a maximum, minimum, or an inflection.

Now let us tackle an actual application! While the definition of the gradient can be applied to a function of three variables, it is much easier to visualize what is going on for functions of two variables. Recall our pictures previously discussed in this problem.

The height of a certain hill (in feet) is given by the function

$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

where y is the distance (in miles) north and x is the distance (in miles) east of the Washington Monument.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point one mile north and one mile east of the Washington Monument?

(d) In what direction is the slope steepest, at that point?

Problem 4

Find the gradient of the following scalar field

$$v(x,y,z) = e^{-z} \sin 2x \cosh y$$

Problem 5

Find the gradient of the following scalar field

$$u(\rho,\theta,z) = \rho^2 z \cos 2\phi$$

Problem 6

Find the gradient of the following scalar field

$$w(r,\theta,\phi) = 10\,r\,\sin^2\theta\cos\phi$$

Problem 7

Vector functions or vector fields are going to be very important in this course. In fact the electric field is an example of a vector function. A vector field $\vec{F}$ can be composed into its components

$$\vec{F} = F_x(x,y,z)\,\hat{\imath} + F_y(x,y,z)\hat{\jmath} + F_z(x,y,z)\,\hat{k}$$

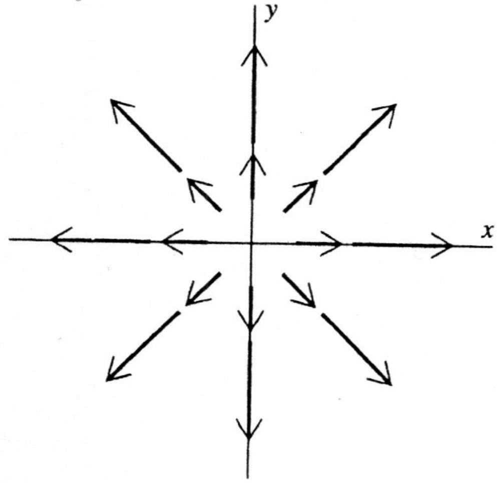
so that one can define the divergence of a vector field $\vec{F}$ as

$$\nabla\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Before we go through some problems let us spend a minute to discuss what the divergence of a vector field actually means physically. Suppose we have the vector field $\vec{F}$

$$\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$$

as illustrated below where we only show a few points for the vector field



For this vector field $\vec{F}$

$$\nabla \cdot \vec{F} = 2$$

which is positive. This positive result means that if we select a point in the vector field there is a net "outflow" in the neighborhood of the field. The divergence tells us how much the vector field spreads out from a given point. The vector field acts as a "source" or "faucet".

Please sketch the following vector field $\vec{G}$

$$\vec{G} = -x\,\hat{\imath} - y\,\hat{\jmath}$$

For this vector field $\vec{G}$

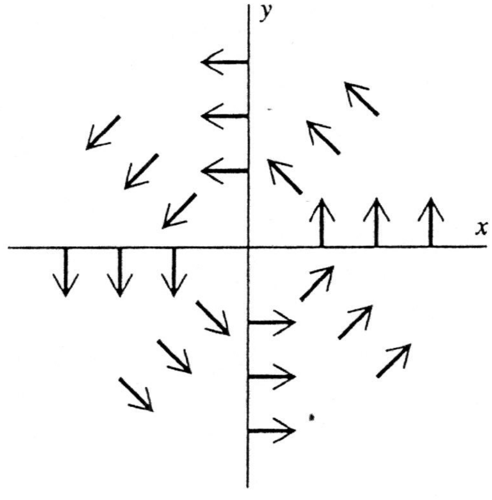
$$\nabla \cdot \vec{G} = -2$$

which is negative. This negative result means that if we select a point in the vector field there is a net "inflow" in the neighborhood of the field. The divergence tells us how much the vector field spreads into a given point. The vector field acts as a "sink" or "drain".

Suppose we have the vector field $\vec{H}$

$$\vec{H} = \frac{-y}{\sqrt{x^2 + y^2}}\,\hat{\imath} + \frac{x}{\sqrt{x^2 + y^2}}\,\hat{\jmath}$$

where some of the vectors of the field are illustrated below

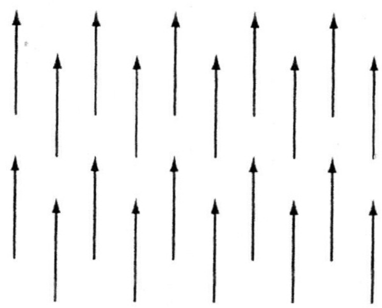


Find the divergence of $\vec{H}$. Explain your answer.

Finally consider the vector field $\vec{J}$

$$\vec{J} = \hat{k}$$

as illustrated below where we only show a few points for the vector field



Find the divergence of $\nabla \cdot \vec{J}$. Explain your answer.

Problem 8

Find the divergence of the following vector field

$$\vec{P} = \hat{\imath}\, x^2 yz + \hat{k}\, xz$$

Problem 9

Find the divergence of the following vector field which is the position vector

$$\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$$

Problem 10

Find the gradient of the norm of the position vector given in Problem 9.

Problem 1

Find the expression for the gradient of a scalar function in cylindrical polar coordinates.

Let us start with

(1-1) $$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Given that

(1-2) $$\rho = (x^2+y^2)^{1/2}$$

(1-3) $$\tan\phi = \frac{y}{x}$$

we can invent a scalar function $f(x,y,z)$

so that

(1-4) $$\vec{\nabla} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Since $f(x,y,z)$ (1-5)

and since
$$\left.\begin{aligned} x &= x(\rho,\phi,z) \\ y &= y(\rho,\phi,z) \\ z &= z(\rho,\phi\, z) \end{aligned}\right\} \quad (1\text{-}6)$$

we can evaluate the following partial derivatives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \tag{1-7}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} \tag{1-8}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial z} \tag{1-9}$$

First of all since z only depends on z and not x, y, these equations reduce to

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} \tag{1-10}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y} \tag{1-11}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z} + \frac{\partial f}{\partial z} \tag{1-12}$$

Next we see from (1-2) and (1-3) that

$$\frac{\partial \rho}{\partial z} = \frac{\partial \phi}{\partial z} = 0 \qquad (1-13)$$

to further get

(1-14) $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

(1-15) $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}$$

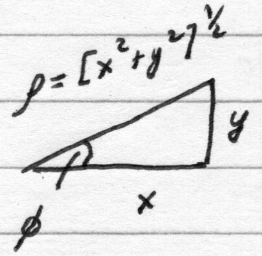
(1-16) $$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z}$$

We need to evaluate these partial derivatives! Using (1-2) we find

(1-17) $$\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\, 2x = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}$$

(1-18) $$\frac{\partial \rho}{\partial y} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\, 2y = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}$$

From (1-3) we can construct a right triangle and the following relations



(1-19) $$\tan\phi = \frac{y}{x}$$

(1-20) $$\sin\phi = \frac{y}{[x^2+y^2]^{1/2}} = \frac{y}{\rho}$$

(1-21) $$\cos\phi = \frac{x}{[x^2+y^2]^{1/2}} = \frac{x}{\rho}$$

so (1-17) and (1-18) become

(1-22) $$\frac{\partial\rho}{\partial x} = \frac{x}{\rho} = \cos\phi$$

(1-23) $$\frac{\partial\rho}{\partial y} = \frac{y}{\rho} = \sin\phi$$

Now what about

$$\frac{\partial\phi}{\partial x} \quad \text{and} \quad \frac{\partial\phi}{\partial y} \;?$$

Let us refer to (1-3)

(1-24) $$y = x\tan\phi$$

and use implicit differentiation

$$\left(\frac{\partial y}{\partial y}\right)_x = x \sec^2\phi \, \frac{\partial \phi}{\partial y} = 1$$

or

$$\frac{\partial \phi}{\partial y} = \frac{1}{x \sec^2\phi}$$

which simplifies to the following using (1-21)

$$(1\text{-}25) \qquad \frac{\partial \phi}{\partial y} = \frac{1}{x\sec^2\phi} = \frac{1}{\rho\cos\phi}\cos^2\phi = \frac{\cos\phi}{\rho}$$

and

$$\left(\frac{\partial y}{\partial x}\right)_y = 0 = \tan\phi + x\sec^2\phi\left(\frac{\partial \phi}{\partial x}\right)$$

or

$$\left(\frac{\partial \phi}{\partial x}\right) = -\frac{\tan\phi}{x\sec^2\phi}$$

Which again using (1-21) simplifies to

$$(1\text{-}26) \qquad \frac{\partial \phi}{\partial x} = \frac{-\sin\phi\cos^2\phi}{\rho\cos\phi\,\cos\phi} = \frac{-\sin\phi}{\rho}$$

Using (1-22), (1-23), (1-25), and (1-26) allow us to recast (1-14), (1-15), and (1-16) as

$$(1\text{-}27) \qquad \frac{\partial f}{\partial x} = \cos\phi \, \frac{\partial f}{\partial \rho} + \left[-\frac{\sin\phi}{\rho} \right] \frac{\partial f}{\partial \phi}$$

$$(1\text{-}28) \qquad \frac{\partial f}{\partial y} = \sin\phi \, \frac{\partial f}{\partial \rho} + \frac{\cos\phi}{\rho} \, \frac{\partial f}{\partial \phi}$$

Plugging (1-27) and (1-28) into (1-1) gives

$$(1\text{-}29) \qquad \vec{\nabla} f = \hat{\imath} \left[\cos\phi \, \frac{\partial f}{\partial \rho} - \frac{\sin\phi}{\rho} \, \frac{\partial f}{\partial \phi} \right]$$

$$+ \hat{\jmath} \left[\sin\phi \, \frac{\partial f}{\partial \rho} + \frac{\cos\phi}{\rho} \, \frac{\partial f}{\partial \phi} \right]$$

$$+ \hat{k} \, \frac{\partial f}{\partial z} \quad \left[\text{This term is obvious!} \right]$$

Next we need to replace $\{ \hat{\imath}, \hat{\jmath}, \hat{k} \}$ by

$$\{ \hat{\rho}, \hat{\phi}, \hat{k} \}$$

Recall from Problem Set I

$$\hat{\imath} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi} \qquad (1\text{-}30)$$

$$\hat{\jmath} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi} \qquad (1\text{-}31)$$

for plane polar coordinates. Clearly for cylindrical polar coordinates

$$\hat{k} = \hat{z} \qquad (1\text{-}32)$$

Plugging (1-30), (1-31), and (1-32) into (1-29) and simplifying we get

$$(1\text{-}33) \qquad \vec{\nabla} f = \left[\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}\right] \times$$

$$\left[\cos\phi\,\frac{\partial f}{\partial \rho} - \frac{\sin\phi}{\rho}\,\frac{\partial f}{\partial \phi}\right] +$$

$$\left[\sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}\right] \times \left[\sin\phi\,\frac{\partial f}{\partial \rho} + \frac{\cos\phi}{\rho}\,\frac{\partial f}{\partial \phi}\right]$$

$$+ \ \hat{z}\,\frac{\partial f}{\partial z}$$

or

$$(1\text{-}34) \qquad \vec{\nabla} f = \cos^2\phi \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{\sin^2\phi}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi}$$

$$\cancelto{0}{-\sin\phi\cos\phi \frac{\partial f}{\partial \rho}\hat{\phi}} - \cancelto{0}{\frac{\sin\phi\cos\phi}{\rho}\frac{\partial f}{\partial \phi}\hat{\rho}}$$

$$+ \sin^2\phi \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{\cos^2\phi}{\rho}\hat{\phi}\frac{\partial f}{\partial \phi} +$$

$$\cancelto{0}{\frac{\sin\phi\,\hat{\rho}\cos\phi}{\rho}\frac{\partial f}{\partial \phi}} + \cancelto{0}{\cos\phi\sin\phi\,\hat{\phi}\frac{\partial f}{\partial \rho}}$$

$$+ \hat{z}\frac{\partial f}{\partial z}$$

or

$$(1\text{-}35) \qquad \vec{\nabla} f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \hat{z}\frac{\partial f}{\partial z}$$

Eliminating f gives the desired result N.B.

This should be $\hat{k}$! Sorry!

$$(1\text{-}36) \qquad \vec{\nabla} = \hat{\rho}\frac{\partial}{\partial \rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

You should have done this derivation at least once in your career. Trust me there is even a more tedious but straightforward way to get the desired result.

Problem 2

Find the expression for the gradient of a scaler function in spherical polar coordinates.

Let us start with

$$(1\text{-}1) \quad \vec{\nabla} = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$$

Given that in spherical polar coordinates

$$(1\text{-}2) \quad r = \sqrt{\rho^2 + z^2} = \sqrt{x^2+y^2+z^2}$$

$$(1\text{-}3) \quad \tan\theta = \frac{[x^2+y^2]^{\frac{1}{2}}}{z}$$

$$(1\text{-}4) \quad \tan\phi = \frac{y}{x}$$

We again can use a scaler function $f(x, y, z)$ such that

$$(1\text{-}5) \quad \vec{\nabla} f = \hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}$$

Since $$f = f(x, y, z) \qquad (1\text{-}6)$$

and

$$\left.\begin{aligned} x &= x(r, \theta, \phi) \\ y &= y(r, \theta, \phi) \\ z &= z(r, \theta, \phi) \end{aligned}\right\} \qquad (1\text{-}7)$$

we can evaluate the following partial derivates

$$(1\text{-}8) \qquad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$(1\text{-}9) \qquad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}$$

$$(1\text{-}10) \qquad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}$$

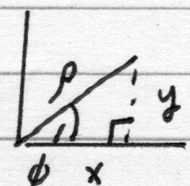
From (1-2) we have

$$(1\text{-}11) \qquad \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$(1\text{-}12) \qquad \frac{\partial r}{\partial y} = \frac{y}{[x^2+y^2+z^2]^{1/2}} = \frac{y}{r}$$

$$(1\text{-}13) \qquad \frac{\partial r}{\partial z} = \frac{z}{[x^2+y^2+z^2]^{1/2}} = \frac{z}{r}$$

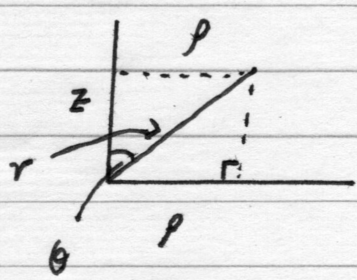
Now if we refer to the geometric picture of spherical polar coordinates we can find some useful relationships



$$\cos\phi = \frac{x}{\rho}$$

$$(1\text{-}14) \qquad x = \rho\cos\phi$$

$$(1\text{-}15) \qquad y = \rho\sin\phi$$



$$(1\text{-}16) \qquad \tan\theta = \frac{\rho}{r}$$

$$(1\text{-}17) \qquad \rho = r\tan\theta$$

$$(1\text{-}18) \qquad \sin\theta = \frac{\rho}{r}$$

$$(1\text{-}19) \qquad \rho = r\sin\theta$$

$$(1\text{-}20) \qquad \cos\theta = \frac{z}{r}$$

$$(1\text{-}21) \qquad z = r\cos\theta$$

Using (1-14) through (1-21) we can recast (1-11), (1-12), and (1-13) as

(1-22) $$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{\rho\cos\phi}{r} = \frac{r\sin\theta\cos\phi}{r}$$

$$= \sin\theta\cos\phi$$

(1-23) $$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{\rho\sin\phi}{r} = \frac{r\sin\theta\sin\phi}{r}$$

$$= \sin\theta\sin\phi$$

(1-24) $$\frac{\partial r}{\partial z} = \frac{z}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

Now let us tackle (1-3)

$$\tan\theta = \frac{[x^2+y^2]^{1/2}}{z}$$

where we need implicit differentiation like in Problem 1 to find the following partial derivatives

$$\frac{\partial\theta}{\partial x},\ \frac{\partial\theta}{\partial y},\ \frac{\partial\theta}{\partial z}$$

Let us start with $\frac{\partial\theta}{\partial x}$

(1-25) $$\sec^2\theta \frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\left[\frac{(x^2+y^2)^{1/2}}{z}\right] = \frac{1}{z}\frac{x}{(x^2+y^2)^{1/2}}$$

or

(1-26) $$\frac{\partial\theta}{\partial x} = \frac{x\cos^2\theta}{z\,(x^2+y^2)^{1/2}}$$

Using (1-14), (1-21), and the definition of ρ from plane polar coordinates this becomes

(1-27) $$\frac{\partial\theta}{\partial x} = \frac{\rho\cos\phi\,\cos^2\theta}{r\cos\theta\;\rho} = \frac{1}{r}\cos\phi\cos\theta$$

Similarly for $\frac{\partial\theta}{\partial y}$

(1-28) $$\frac{\partial\theta}{\partial y} = \frac{y\cos^2\theta}{z\,(x^2+y^2)^{1/2}}$$

and using (1-15), (1-21), and the definition of ρ we get

(1-29) $$\frac{\partial\theta}{\partial y} = \frac{\rho\sin\phi\,\cos^2\theta}{r\cos\theta\;\rho} = \frac{1}{r}\sin\phi\cos\theta$$

Finally, we tackle $\frac{\partial \theta}{\partial z}$

(1-30) $$\frac{\partial}{\partial z}(\tan\theta) = \frac{\partial}{\partial z}\left[\frac{(x^2+y^2)^{\frac{1}{2}}}{z}\right]$$

(1-31) $$\sec^2\theta \frac{\partial \theta}{\partial z} = -\frac{1}{z^2}(x^2+y^2)^{\frac{1}{2}}$$

or

(1-32) $$\frac{\partial \theta}{\partial z} = -\frac{1}{z^2}\cos^2\theta\,(x^2+y^2)^{\frac{1}{2}}$$

Using (1-21) this becomes

(1-33) $$\frac{\partial \theta}{\partial z} = -r^{-2}(x^2+y^2)^{\frac{1}{2}} = -\frac{(x^2+y^2)^{\frac{1}{2}}}{r^2}$$

and from (1-19) and the definition of ρ, we get

(1-34) $$\left(\frac{\partial \theta}{\partial z}\right) = \frac{-\rho}{r^2} = -\frac{r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

Finally, we can use (1-4) and implicit differentiation to obtain the partial derivatives

$$\frac{\partial \phi}{\partial x}, \ \frac{\partial \phi}{\partial y}, \ \frac{\partial \phi}{\partial z}$$

as we did in Problem 1.

$$(1\text{-}35) \qquad y = x \tan \phi$$

$$(1\text{-}36) \qquad \frac{\partial y}{\partial y} = x \sec^2 \phi \ \frac{\partial \phi}{\partial y} = 1$$

or

$$(1\text{-}37) \qquad \frac{\partial \phi}{\partial y} = \frac{1}{x \sec^2 \phi} = \frac{\cos^2 \phi}{x}$$

which according to (1-14) becomes

$$(1\text{-}38) \qquad \frac{\partial \phi}{\partial y} = \frac{\cos^2 \phi}{\rho \cos \phi} = \frac{1}{\rho} \cos \phi$$

Similarly

$$(1\text{-}39) \qquad y = x \tan \phi$$

$$(1\text{-}40) \qquad \frac{\partial y}{\partial x} = \tan \phi + x \sec^2 \phi \ \frac{\partial \phi}{\partial x}$$

(1-41) $$0 = \tan\phi + x\sec^2\phi\,\frac{\partial\phi}{\partial x}$$

(1-42) $$\frac{\partial\phi}{\partial x} = -\frac{\tan\phi}{x\sec^2\phi} = \frac{-\sin\phi\,\cos^2\phi}{x\cos\phi}$$

(1-43) $$\frac{\partial\phi}{\partial x} = -\frac{1}{x}\sin\phi\,\cos\phi$$

Finally, let us utilize (1-14) to obtain

(1-44) $$\frac{\partial\phi}{\partial x} = -\frac{1}{\rho}\,\frac{\sin\phi\cos\phi}{\cos\phi} = -\frac{\sin\phi}{\rho}$$

The last partial derivative we need is $\frac{\partial\phi}{\partial z}$

(1-45) $$\frac{\partial}{\partial z}[y] = \frac{\partial}{\partial z}[x\tan\phi]$$

which clearly vanishes!

Now let us return to (1-8), (1-9), and (1-10)

(1-46) $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

(1-47) $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}$$

(1-48) $$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}$$

Using (1-22), (1-23), (1-24), (1-27), (1-29), (1-34), (1-38) and (1-44), these equations become

(1-49) $$\frac{\partial f}{\partial x} = \sin\theta\cos\phi\,\frac{\partial f}{\partial r} + \frac{1}{r}\cos\phi\cos\theta\,\frac{\partial f}{\partial \theta} - \frac{\sin\phi}{\rho}\frac{\partial f}{\partial \phi}$$

(1-50) $$\frac{\partial f}{\partial y} = \sin\theta\sin\phi\,\frac{\partial f}{\partial r} + \frac{1}{r}\sin\phi\cos\theta\,\frac{\partial f}{\partial \theta} + \frac{\cos\phi}{\rho}\frac{\partial f}{\partial \phi}$$

(1-51) $$\cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r} \frac{\partial f}{\partial \theta}$$

and (1-5) becomes

(1-52) $$\vec{\nabla} f = \hat{i}\left[\sin\theta\cos\phi \frac{\partial f}{\partial r} + \frac{1}{r}\cos\phi\cos\theta\frac{\partial f}{\partial\theta} - \frac{\sin\phi}{\rho}\frac{\partial f}{\partial\phi}\right]$$

$$+ \hat{j}\left[\sin\theta\sin\phi\frac{\partial f}{\partial r} + \frac{1}{r}\sin\phi\cos\theta\frac{\partial f}{\partial\theta} + \frac{\cos\phi}{\rho}\frac{\partial f}{\partial\phi}\right]$$

$$+ \hat{k}\left[\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right]$$

Now we use the same trick as performed in Problem Set 1. From Problem Set I we have

(1-53) $$\hat{i} = \sin\theta\cos\phi\ \hat{r} + \cos\theta\cos\phi\ \hat{\theta} - \sin\phi\ \hat{\phi}$$

(1-54) $$\hat{j} = \sin\theta\sin\phi\ \hat{r} + \cos\theta\sin\phi\ \hat{\theta} + \cos\phi\ \hat{\phi}$$

(1-55) $$\hat{k} = \cos\theta\ \hat{r} - \sin\theta\ \hat{\theta}$$

If we substitute (1-53), (1-54), and (1-55) into (1-52) we obtain

(1-56) $$\vec{\nabla}f = \left[\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right]\left[\sin\theta\cos\phi\frac{\partial f}{\partial r} + \frac{1}{r}\cos\phi\cos\theta\frac{\partial f}{\partial\theta} - \frac{\sin\phi}{\rho}\frac{\partial f}{\partial\phi}\right]$$

$$+ \left[\sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}\right] \times \left[\sin\theta\sin\phi\frac{\partial f}{\partial r} + \frac{1}{r}\sin\phi\cos\theta\frac{\partial f}{\partial\theta} + \frac{\cos\phi}{\rho}\frac{\partial f}{\partial\phi}\right]$$

$$+ \left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]\left[\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right]$$

Now carefully expanding this out

$$(1\text{-}57)\quad \vec{\nabla} f = \overset{(1)}{\sin^2\theta \cos^2\phi\, \hat{r} \frac{\partial f}{\partial r}} + \overset{(2)}{\frac{1}{r} \cos^2\phi \sin\theta \cos\theta\, \hat{r} \frac{\partial f}{\partial \theta}}$$

$$- \overset{(3)}{\frac{\sin\phi}{\rho} \sin\theta \cos\phi\, \hat{r} \frac{\partial f}{\partial \phi}} + \overset{(4)}{\cos\theta \sin\theta\, \hat{\theta} \cos^2\phi \frac{\partial f}{\partial r}}$$

$$+ \overset{(5)}{\frac{1}{r} \cos^2\phi \cos^2\theta\, \hat{\theta} \frac{\partial f}{\partial \theta}} - \overset{(6)}{\frac{\sin\phi}{\rho} \cos\theta \cos\phi\, \hat{\theta} \frac{\partial f}{\partial \phi}}$$

$$- \overset{(7)}{\sin\phi \cos\phi \sin\theta\, \hat{\phi} \frac{\partial f}{\partial r}} - \overset{(8)}{\frac{1}{r} \sin\phi \cos\phi \cos\theta\, \hat{\phi} \frac{\partial f}{\partial \theta}}$$

$$+ \overset{(9)}{\frac{\sin^2\phi}{\rho}\, \hat{\phi} \frac{\partial f}{\partial \phi}} + \overset{(10)}{\sin^2\theta \sin^2\phi\, \hat{r} \frac{\partial f}{\partial r}}$$

$$+ \overset{(11)}{\frac{1}{r} \sin^2\phi \sin\theta \cos\theta\, \hat{r} \frac{\partial f}{\partial \theta}} +$$

$$\overset{(12)}{\frac{\sin\theta \sin\phi \cos\phi}{\rho}\, \hat{r} \frac{\partial f}{\partial \phi}} + \underset{(13)}{\cos\theta \sin\theta \sin^2\phi\, \hat{\theta} \frac{\partial f}{\partial r}}$$

$$+ \underset{(14)}{\frac{1}{r} \sin\phi \cos^2\theta \sin\phi\, \hat{\theta} \frac{\partial f}{\partial \theta}} + \overset{(15)}{\frac{\cos\theta \cos\phi \sin\phi}{\rho}\, \hat{\theta} \frac{\partial f}{\partial \phi}}$$

$$+ \underset{(16)}{\sin\theta \cos\phi \sin\phi\, \hat{\phi} \frac{\partial f}{\partial r}} + \underset{(17)}{\frac{1}{r} \sin\phi \cos\theta \cos\phi\, \hat{\phi} \frac{\partial f}{\partial \theta}} + \underset{(18)}{\frac{\cos^2\phi}{\rho}\, \hat{\phi} \frac{\partial f}{\partial \phi}}$$

$$+ \cos^2\theta \, \hat{r} \, \frac{\partial f}{\partial r} \quad - \sin\theta \cos\theta \, \hat{\theta} \, \frac{\partial f}{\partial r}$$

(19) (20)

$$-\frac{\sin\theta\cos\theta}{r} \, \hat{r} \, \frac{\partial f}{\partial \theta} \quad + \frac{\sin^2\theta}{r} \, \frac{\partial f}{\partial \theta} \, \hat{\theta}$$

(21) (22)

We have numbered all of our terms so that we can see how things simplify

(3) + (12) $\longrightarrow 0$

(6) + (15) $\longrightarrow 0$

(7) + (16) $\longrightarrow 0$

(8) + (17) $\longrightarrow 0$

Let us combine (9) + (18) to yield

(1-58) $$\frac{\sin^2\phi}{\rho} \, \hat{\phi} \, \frac{\partial f}{\partial \phi} + \frac{\cos^2\phi}{\rho} \, \hat{\phi} \, \frac{\partial f}{\partial \phi} = \frac{1}{\rho} \, \hat{\phi} \, \frac{\partial f}{\partial \phi}$$

Let us combine (1) + (10) to yield

(1-59) $$\sin^2\theta \cos^2\phi \, \hat{r} \, \frac{\partial f}{\partial r} + \sin^2\theta \sin^2\phi \, \hat{r} \, \frac{\partial f}{\partial r}$$

$$= \sin^2\theta \, \hat{r} \, \frac{\partial f}{\partial r}$$

and then combine (1-59) with (19) to yield

(1-60) $$\sin^2\theta\, \hat{r}\, \frac{\partial f}{\partial r} + \cos^2\theta\, \hat{r}\, \frac{\partial f}{\partial r} = \hat{r}\, \frac{\partial f}{\partial r}$$

Let us combine (5) with (14) to get

(1-61) $$\frac{1}{r} \cos^2\phi \cos^2\theta\, \hat{\theta}\, \frac{\partial f}{\partial \theta} + \frac{1}{r} \sin^2\phi \cos^2\theta\, \hat{\theta}\, \frac{\partial f}{\partial \theta}$$

$$= \frac{1}{r} \cos^2\theta\, \hat{\theta}\, \frac{\partial f}{\partial \theta}$$

Let us combine (4) with (13) to get

(1-62) $$\cos\theta \sin\theta\, \hat{\theta} \cos^2\phi\, \frac{\partial f}{\partial r} + \cos\theta \sin\theta \sin^2\phi\, \hat{\theta}\, \frac{\partial f}{\partial r}$$

$$= \cos\theta \sin\theta\, \hat{\theta}\, \frac{\partial f}{\partial r}$$

Let us combine (2) with (11) to yield

(1-63) $$\frac{1}{r} \cos^2\phi \sin\theta \cos\theta\, \hat{r}\, \frac{\partial f}{\partial \theta} + \frac{1}{r} \sin^2\phi \sin\theta \cos\theta\, \hat{r}\, \frac{\partial f}{\partial \theta}$$

$$= \frac{1}{r} \sin\theta \cos\theta\, \hat{r}\, \frac{\partial f}{\partial \theta}$$

Let us combine (1-62) with 20 to get

(1-64) $$\cos\theta \sin\theta \,\hat{\theta}\, \frac{\partial f}{\partial r} - \sin\theta\cos\theta\,\hat{\theta}\,\frac{\partial f}{\partial r} = 0$$

Let us combine (1-63) with 21 to get

(1-65) $$\frac{1}{r}\sin\theta\cos\theta\,\hat{r}\,\frac{\partial f}{\partial \theta} - \frac{1}{r}\sin\theta\cos\theta\,\hat{r}\,\frac{\partial f}{\partial \theta} = 0$$

Finally, let us combine (1-61) with 22 to yield

(1-66) $$\frac{1}{r}\cos^2\theta\,\hat{\theta}\,\frac{\partial f}{\partial \theta} + \frac{1}{r}\sin^2\theta\,\hat{\theta}\,\frac{\partial f}{\partial \theta}$$

$$= \frac{1}{r}\hat{\theta}\,\frac{\partial f}{\partial \theta}$$

Using (1-58), (1-60), (1-66), Eq.(1-57) becomes

(1-67) $$\vec{\nabla} f = \frac{1}{\rho}\hat{\phi}\,\frac{\partial f}{\partial \phi} + \hat{r}\,\frac{\partial f}{\partial r} + \frac{1}{r}\hat{\theta}\,\frac{\partial f}{\partial \theta}$$

or since $\rho = r\sin\theta$ from (1-18)

(1-68) $$\vec{\nabla} f = \hat{r}\,\frac{\partial f}{\partial r} + \frac{1}{r}\hat{\theta}\,\frac{\partial f}{\partial \theta} + \frac{1}{r\sin\theta}\hat{\phi}\,\frac{\partial f}{\partial \phi}$$

or

(1-69) $$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\hat{\phi}\frac{\partial}{\partial \phi}$$

Problem 3

(a) For

$$h(x,y) = 10\,(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

in feet

where y is the distance (in miles) north, and x is the distance (in miles) east of the Washington Monument.

Where is the top of the hill located?

$$\vec{\nabla} h = \hat{i}\,\frac{\partial h}{\partial x} + \hat{j}\,\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\left[20xy - 30x^2 - 40y^2 - 180x + 280y + 120\right]$$

$$\frac{\partial h}{\partial x} = 20y - 60x - 180$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left[20xy - 30x^2 - 40y^2 - 180x + 280y + 120\right]$$

$$\frac{\partial h}{\partial y} = 20x - 80y + 280$$

$$\vec{\nabla} h = \hat{i}\,(20y - 60x - 180) + \hat{j}\,(20x - 80y + 280)$$

The top of the hill is located at

$$\vec{\nabla} h = 0$$

so

$$20y - 60x - 180 = 0$$
$$20x - 80y + 280 = 0$$

Let us solve for x, y as follows

$$4 \cdot 20y - 60 \cdot 4x - 180(4) = 0$$
$$20x - 80y + 280 = 0$$

$$80y - 240x - 720 = 0$$
$$20x - 80y + 280 = 0$$

$$-220x - (440) = 0$$

$$x = -2 \text{ miles}$$

$$-40 - 80y + 280 = 0$$
$$-80y + 240 = 0$$

$$y = \frac{240}{80} = 3 \text{ miles}$$

The top of the hill is located at $(-2, 3)$ in units of miles

-30-

Maximize 20xy -30 x^2 -40 y^2 -180 x +280 y +120

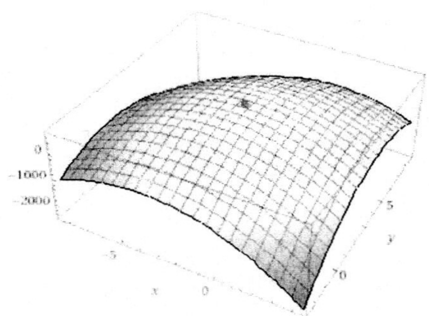
Input interpretation:

maximize	$20xy - 30x^2 - 40y^2 - 180x + 280y + 120$

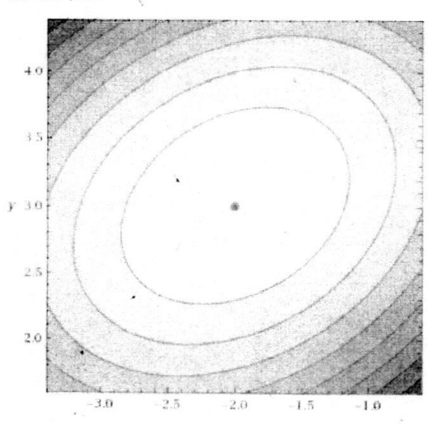
Global maximum:

$\max\{20xy - 30x^2 - 40y^2 - 180x + 280y + 120\} = 720$ at $(x, y) = (-2, 3)$

3D plot:



Contour plot:



Related Queries:

(b) How high is the hill?

$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

At $x = -2$, $y = 3$

$$h(x,y) = 10[2(-6) - 3\cdot 4 - 4\cdot 9 + 36 + 28\cdot 3 + 12]$$

$$= 10[-12 - 12 - 36 + 36 + 84 + 12]$$

$$= 10[-24 + 96] = 10[72] = 720 \text{ ~~miles~~ feet}$$

Clearly $h(x,y)$ [in ~~miles~~ feet] has scaled in a conversion factor

(c) How steep is the slope (in feet per mile) at a point one mile north and one mile east of the Washington Monument?

If $\vec{\nabla} h$ is the gradient of h the

$\|\vec{\nabla} h\|$ is the slope of the hill at some point

$$\|\vec{\nabla} h\| = \left[(20y - 60x - 180)^2 + (20x - 80y + 280)^2\right]^{1/2}$$

For $y = 1$ and $x = 1$

$$\|\vec{\nabla} h\| = \left[(20 - 60 - 180)^2 + (20 - 80 + 280)^2\right]^{1/2}$$

$$\|\vec{\nabla} h\| = \left[(-240 + 20)^2 + (300 - 80)^2\right]^{1/2}$$

$$\|\vec{\nabla} h\| = \left[(-220)^2 + (220)^2\right]^{1/2}$$

$$\|\vec{\nabla} h\| = \left[(220)(220) \cdot 2\right]^{1/2} = \left[96,800\right]^{1/2}$$

$\|\vec{\nabla} h\|$ is in units of $\frac{\text{feet}}{\text{mile}}$

$$\|\vec{\nabla} h\| = 220\sqrt{2} \text{ feet mile}^{-1}$$

(d) In what direction is the slope steepest, at that point

$$\vec{\nabla} h = \hat{i}\,(20y - 60x - 180) + \hat{j}\,(20x - 80y + 280)$$

At the point $y = 1$, $x = 1$, $\vec{\nabla} h$ becomes

$$\vec{\nabla} h = \hat{i}\,(20 - 60 - 180) + \hat{j}\,(20 - 80 + 280)$$

$$\vec{\nabla} h = -\hat{i}\,220 + \hat{j}\,220$$ (This is the direction that the slope is steepest)

The gradient of h points in the direction of steepest increase!

Problem 4.

$$V(x,y,z) = e^{-z} \sin 2x \cosh y$$

$$\vec{\nabla} V = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$$

$$= \hat{i}\ 2e^{-z}\cos 2x \cosh y + \hat{j} e^{-z} \sin 2x \sinh y$$

$$- \hat{k} e^{-z} \sin 2x \cosh y$$

Problem 5

$$U(\rho, \theta, z) = \rho^2 z \cos 2\phi$$

$$\vec{\nabla} U = \hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}$$ [NOT A SMART CHOICE OF COORDINATES]

$$= \hat{\rho}\frac{\partial U}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial U}{\partial \phi} + \hat{k}\frac{\partial U}{\partial z}$$ [SMARTER CHOICE!]

$$= \hat{\rho}\ 2\rho z \cos 2\phi + \hat{\phi}\frac{\rho^2 z}{\rho}(-\sin 2\phi) 2 + \hat{k}\rho^2 \cos 2\phi$$

$$= \hat{\rho}\ 2\rho z \cos 2\phi - \hat{\phi}\ 2\rho z \sin 2\phi$$

$$+ \hat{k}\rho^2 \cos 2\phi$$

Problem 6

$$\vec{\nabla} W = \vec{\nabla}\,(10 r \sin^2\theta \cos\phi)$$

Let us utilize the gradient operator in spherical polar coordinates

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\vec{\nabla} W = \hat{r}\frac{\partial}{\partial r}[10 r \sin^2\theta \cos\phi] + \frac{\hat{\theta}}{\sin\theta}\frac{1}{r}\frac{\partial}{\partial\theta}[10 r \sin^2\theta \cos\phi]$$

$$+ \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}[10 r \sin^2\theta \cos\phi]$$

$$\vec{\nabla} W = \hat{r}\, 10 \sin^2\theta \cos\phi + \hat{\theta}\, 10 \sin 2\theta \cos\phi$$

$$+ \left(-\frac{1}{r\sin\theta} r \hat{\phi} \sin^2\theta \sin\phi\right)$$

$$\vec{\nabla} W = \hat{r}\, 10 \sin^2\theta \cos\phi + \hat{\theta}\, 10 \sin 2\theta \cos\phi$$

$$-10\hat{\phi} \sin\theta \sin\phi$$

N.B. $y = \sin^2\theta$

$v = \sin\theta \qquad dv = \cos\theta\, d\theta$

$y = v^2 \qquad dy = 2v\,dv$

$dy = 2\sin\theta\cos\theta\, d\theta$

$\dfrac{dy}{d\theta} = \sin 2\theta \quad \leftarrow \quad dy = \dfrac{2}{2}\sin 2\theta\, d\theta$

Problem 7

$$\vec{H} = \frac{-y}{(x^2+y^2)^{1/2}}\hat{i} + \frac{x}{(x^2+y^2)^{1/2}}\hat{j}$$

$$\vec{\nabla}\cdot\vec{H} = \frac{\partial}{\partial x}\left[\frac{-y}{(x^2+y^2)^{1/2}}\right] + \frac{\partial}{\partial y}\left[\frac{x}{(x^2+y^2)^{1/2}}\right]$$

$$\vec{\nabla}\cdot\vec{H} = \frac{+y\,\frac{1}{2}\,(x^2+y^2)^{-1/2}}{(x^2+y^2)}(2x)$$

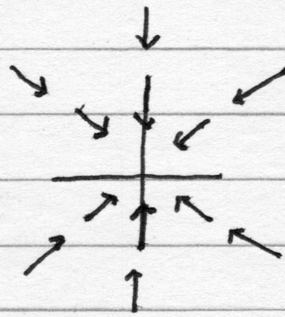
$$+ \frac{-x\,\frac{1}{2}\,(x^2+y^2)^{-1/2}}{(x^2+y^2)}(2y)$$

$$= 0$$

The flow into a small region of volume (or point) is equal to the flow outward!

$\vec{G}$ (sketch)

$$\vec{G} = -\hat{i}x - \hat{j}y$$



Problem 7 (con't)

$$\vec{J} = \hat{k}$$

$$\vec{\nabla} = \cancel{\frac{\partial J_x}{\partial x}}^{0} + \cancel{\frac{\partial J_y}{\partial y}}^{0} + \frac{\partial J_z}{\partial z} = \frac{\partial (1)}{\partial z} = 0$$

The flow into a small region of volume (or point) is equal to the flow outwart!

Problem 8

$$\vec{P} = x^2yz\,\hat{i} + xz\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

$$= 2xyz + 0 + x$$

$$\text{div}\,\vec{P} = 2xyz + x$$

Problem 9

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial y} + \frac{\partial r_z}{\partial z}$$

$$= 1 + 1 + 1 = 3$$

Problem 10

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$

$$r = |\vec{r}| = (x^2 + y^2 + z^2)^{1/2}$$

$$\vec{\nabla}r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$\vec{\nabla}r = \hat{i}\,\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2x$$

$$+\hat{j}\,\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2y$$

$$+\hat{k}\,\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2z$$

$$\vec{\nabla} r = \frac{\hat{i}\,x}{(x^2+y^2+z^2)^{1/2}} + \frac{\hat{j}\,y}{(x^2+y^2+z^2)^{1/2}} + \frac{\hat{k}\,z}{(x^2+y^2+z^2)^{1/2}}$$

$$\vec{\nabla} r = \frac{\hat{i}x + \hat{j}y + \hat{k}z}{(x^2+y^2+z^2)^{1/2}} = \frac{\vec{r}}{r} = \hat{r}$$

Does this make? It says that the distance from the origin increases must rapidly in the radial direction and that this rate of increase in that direction is 1. This is what you would expect!