USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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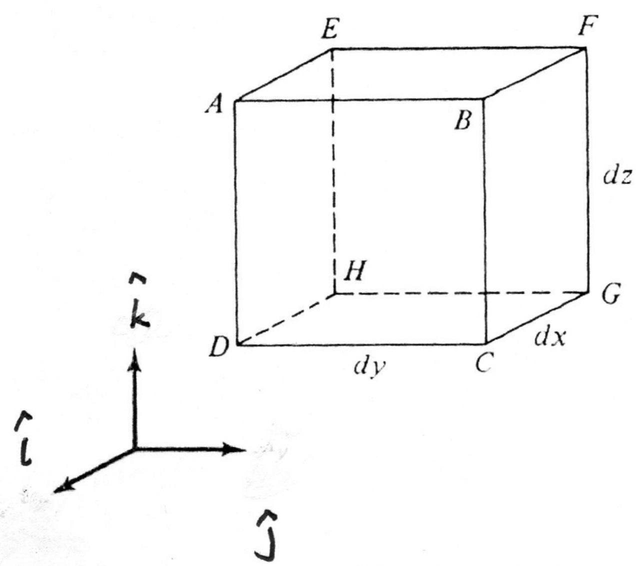
PROBLEM SET III
(due Friday, July 10, 2020)

Problem 1

Here is a picture of the differential volume element in Cartesian coordinates

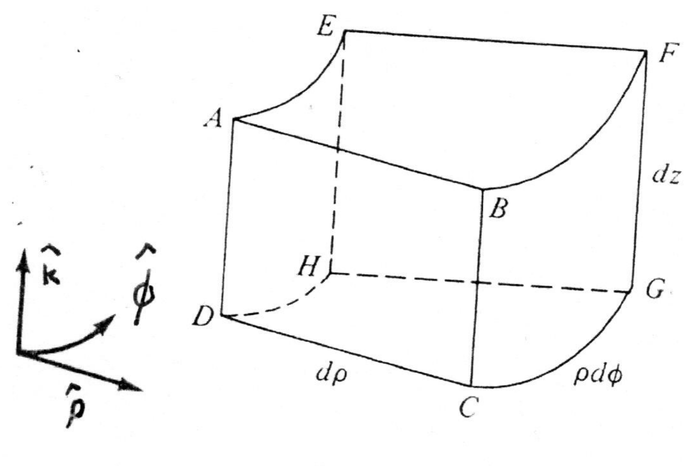


Using this figure match the items in the left column with those in the right column.

(a) $d\vec{r}$ *from* A *to* B	(i) $dy\,dz\,\hat{\imath}$
(b) $d\vec{r}$ *from* A *to* D	(ii) $-\,dx\,dz\,\hat{\jmath}$
(c) $d\vec{r}$ *from* A *to* E	(iii) $dx\,dy\,\hat{k}$
(d) $d\vec{S}$ *for face* $ABCD$	(iv) $-\,dx\,dy\,\hat{k}$
(e) $d\vec{S}$ *for face* $AEHD$	(v) $-\,dx\,\hat{\imath}$
(f) $d\vec{S}$ *for face* $DCGH$	(vi) $dy\,\hat{\jmath}$
(g) $d\vec{S}$ *for face* $ABFE$	(vii) $-\,dz\,\hat{k}$

Problem 2

Here is a picture of the differential volume element in cylindrical polar coordinates. You should realize that although it does not look like a box as in the previous problem it becomes one for the small differentials used. Please appreciate the beauty of the differential in calculus!

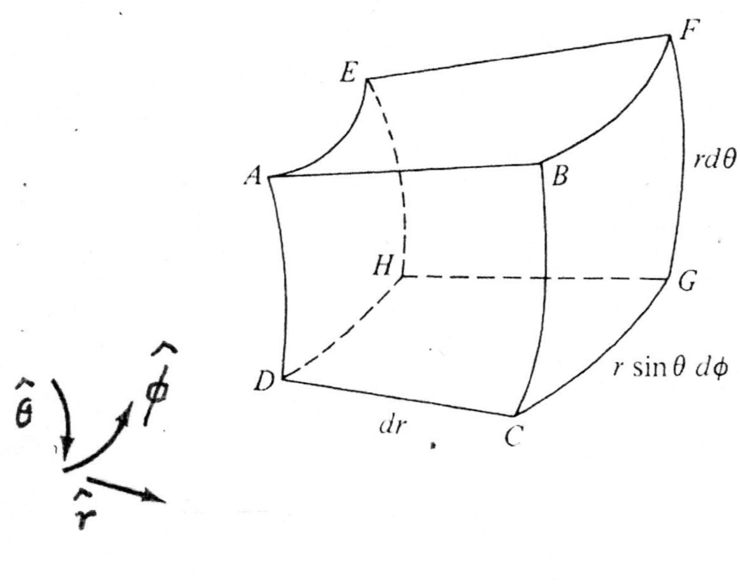


Using this figure match the items in the left column with those in the right column.

(a) $d\vec{r}$ *from* E *to* A	(i) $-\rho\, d\phi\, dz\, \hat{\rho}$
(b) $d\vec{r}$ *from* B *to* A	(ii) $-d\rho\, dz\, \hat{\phi}$
(c) $d\vec{r}$ *from* D *to* A	(iii) $-\rho\, d\rho\, d\phi\, \hat{k}$
(d) $d\vec{S}$ *for face* $ABCD$	(iv) $\rho\, d\rho\, d\phi\, \hat{k}$
(e) $d\vec{S}$ *for face* $AEHD$	(v) $-d\rho\, \hat{\rho}$
(f) $d\vec{S}$ *for face* $ABFE$	(vi) $-\rho\, d\phi\, \hat{\phi}$
(g) $d\vec{S}$ *for face* $DCGH$	(vii) $dz\, \hat{k}$

Problem 3

Here is a picture of the differential volume element in spherical polar coordinates. You should realize that although it does not look like a box as in the Problem 1 it becomes one for the small differentials used. Please appreciate the beauty of the differential in calculus!



Using this figure match the items in the left column with those in the right column.

$(a)\ d\vec{r}\ \ from\ \ A\ \ to\ \ D$	$(i)\ \ -r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$
$(b)\ d\vec{r}\ \ from\ \ E\ \ to\ \ A$	$(ii)\ \ -r\ \sin\theta\, dr\ d\phi\, \hat{\theta}$
$(c)\ d\vec{r}\ \ from\ \ A\ \ to\ \ B$	$(iii)\ \ r\, dr\, d\theta\, \hat{\phi}$
$(d)\ d\vec{S}\ \ for\ \ face\ \ EFGH$	$(iv)\ \ dr\, \hat{r}$
$(e)\ d\vec{S}\ \ for\ \ face\ \ AEHD$	$(v)\ \ r\, d\theta\, \hat{\theta}$
$(f)\ d\vec{S}\ \ for\ \ face\ \ ABFE$	$(vi)\ \ -r\ \sin\theta\, d\phi\, \hat{\phi}$

Problem 4

Using the figures in Problems 1, 2, and 3, prove to yourself that the following expressions are true. It is very important that you see this!

(4-1) The displacement vectors or differential differentials in Cartesian, cylindrical polar, and spherical polar coordinates are

$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$$

$$d\vec{r} = d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{k}$$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$$

(4-2) The differential surface elements in Cartesian, cylindrical polar, and spherical polar coordinates are

$$d\vec{S} = dy\,dz\,\hat{\imath} + dx\,dz\,\hat{\jmath} + dx\,dy\,\hat{k}$$

$$d\vec{S} = \rho\,d\phi\,dz\hat{\rho} + d\rho\,dz\,\hat{\phi} + \rho\,d\rho\,d\phi\,\hat{k}$$

$$d\vec{S} = r^2 \sin\theta\,d\theta\,d\phi\,\hat{r} + r\,\sin\theta\,d\phi\,dr\,\hat{\theta} + r\,dr\,d\theta\,\hat{\phi}$$

(4-3) The differential volume elements in Cartesian, cylindrical polar, and spherical polar coordinates are

$$dV = dx\,dy\,dz$$

$$dV = \rho\,d\rho\,d\theta\,dz$$

$$dV = r^2 \sin\theta\,dr\,d\theta\,d\phi$$

Problem 5

Evaluate the following surface integral

$$\oint_S \vec{A}\cdot d\vec{S}$$

where

$$\vec{A} = 4xz\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}$$

over the unit cube described by

$$0 \le x \le 1$$

$$0 \le y \le 1$$

$$0 \le z \le 1$$

Problem 6

Determine the flux of $\vec{G}$

$$\oint_S \vec{G} \cdot d\vec{S}$$

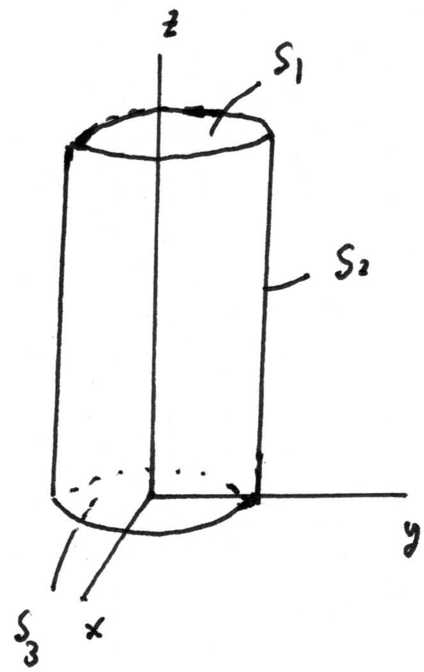
where

$$\vec{G}(r) = 10e^{-2z}(\rho\,\hat{\rho}_z + \hat{k})$$

out of the entire surface of the cylinder

$$\rho = 1$$

$$0 \leq z \leq 1$$



$$S = S_1 + S_2 + S_3$$

Problem 7

Determine the flux of $\vec{D}$

$$\oint_S \vec{D} \cdot d\vec{S}$$

where

$$\vec{D} = \rho^2 \cos^2 \phi \, \hat{\rho} + z \, \sin \phi \, \hat{\phi}$$

over the closed surface of the cylinder

$$\rho = 4$$

$$0 \leq z \leq 1$$

Problem 8

If

$$g(\theta, \phi) = r^2$$

evaluate

$$\int_V g(\theta, \phi) \, dV$$

over the hemisphere of radius 1 centered at the origin where

$$\rho = 4$$

$$z \geq 0$$

$$g(\theta, \phi) = r^2$$

Problem 9

Evaluate the line integral

$$\oint_S \vec{A} \cdot d\vec{r}$$

where

$$\vec{A} = \rho \cos\phi \, \hat{\rho} + z \, \sin\phi \, \hat{k}$$

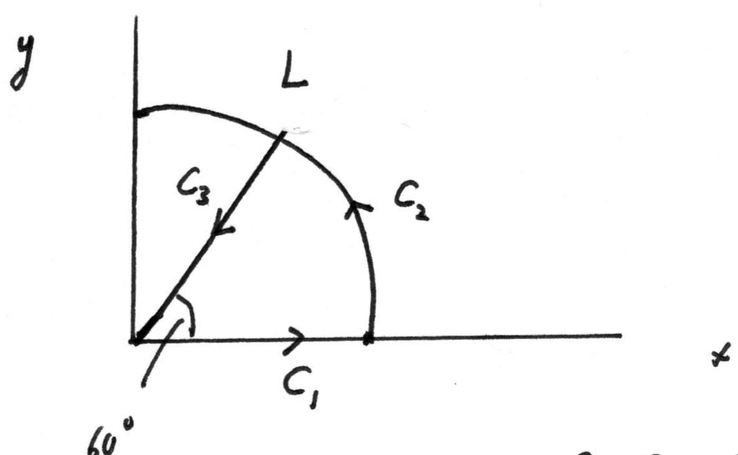
and C is the edge L of the wedge defined by

$$0 \le \rho \le 2$$

$$0 \le \phi \le 60^o$$

$$z = 0$$

as shown in the figure below



$L = C = C_1 + C_2 + C_3$

Problem 10

1. Error in Problem Set I (Solutions): On Page 3 you should have

$$\| \vec{u} \|^2 = \| \vec{v} \|^2$$

2. In the taped version of Lecture 1, I misspelled "Cartesian".

Problem 1

(a) $\longrightarrow$ $dy\,\hat{j}$ (vi) ✓

(b) $\longrightarrow$ $-dz\,\hat{k}$ (vii) ✓

(c) $\longrightarrow$ $-dx\,\hat{i}$ (v) ✓

(d) $\longrightarrow$ $dy\,dz\,(\hat{i})$ (i) ✓

(e) $\longrightarrow$ $dx\,dz\,(-\hat{j})$ (ii) ✓

(f) $\longrightarrow$ $-dx\,dy\,(\hat{k})$ (iv) ✓

(g) $\longrightarrow$ $dy\,dx\,(\hat{k})$ (iii) ✓

Problem 2

(a) $\longrightarrow$ $-\ \rho\, d\phi\, \hat{\phi}$ (vi) ✓

(b) $\longrightarrow$ $-d\rho\ \hat{\rho}$ (v) ✓

(c) $\rightarrow +dz\ \hat{k}$ (vii) ✓

(d) $\rightarrow -dz\, d\rho\, \hat{\phi}$ (ii) ✓

(e) $\longrightarrow$ $-\rho\, d\phi\, dz\, \hat{\rho}$ (i) ✓

(f) $\longrightarrow \rho\, d\rho\, d\phi\ \hat{k}$ (iv) ✓

(g) $\rightarrow -\rho\, d\rho\, d\phi\ \hat{k}$ (iii) ✓

Problem 3 (Remember consulting attached reference from McQuarrie and Simons to be sure you understand dV for spherical polar coordinates!)

(a) $\longrightarrow \quad + \, r\,d\theta\,\hat{\theta}$ (v)

(b) $\longrightarrow \quad -r\sin\theta\,d\phi\,\hat{\phi}$ (vi)

(c) $\longrightarrow \quad dr\,\hat{r}$ (iv)

(d) $\longrightarrow \quad r\,dr\,d\theta\,\hat{\phi}$ (iii)

(e) $\longrightarrow \quad -r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$ (i)

(f) $\longrightarrow \quad -r\,dr\sin\theta\,d\phi\,\hat{\theta}$ (ii)

Problem 4

(4-1) True by inspection

(4-2) True by inspection

(4-3) True by inspection

Problem 5

Evaluate the following surface integral

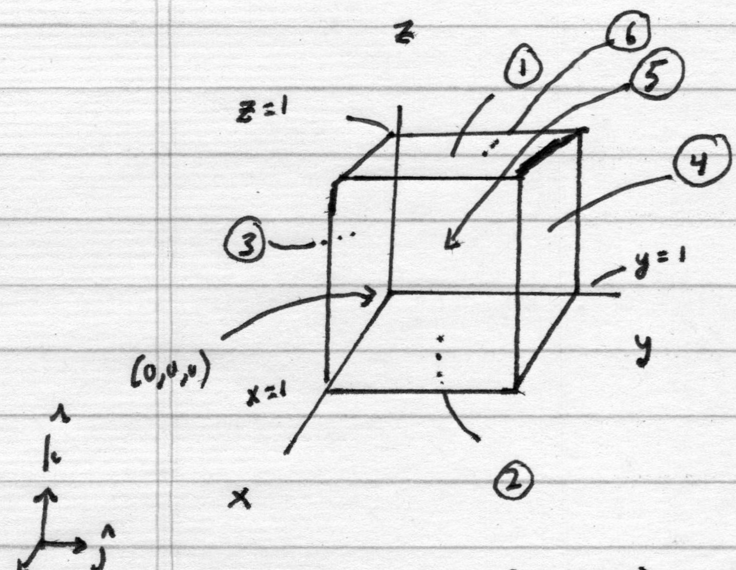
$$\oint_S \vec{A} \cdot d\vec{S}$$

Where

$$\vec{A} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$$

over the unit cube described by

$$0 \le x \le 1$$
$$0 \le y \le 1$$
$$0 \le z \le 1$$



Top S_1

Bottom S_2

Left S_3

Right S_4

Front S_5

Back S_6

$$\oint_S \vec{A} \cdot d\vec{S} = \int_{S_1} \vec{A} \cdot d\vec{S} + \int_{2} \vec{A} \cdot d\vec{S} + \int_{3} \vec{A} \cdot d\vec{S}$$

$$+ \int_{S_4} \vec{A} \cdot d\vec{S} + \int_{S_5} \vec{A} \cdot d\vec{S} + \int_{S_6} \vec{A} \cdot d\vec{S}$$

$$\int_{S_1} \vec{A}\cdot d\vec{S} \qquad d\vec{S} = \hat{k}\, dx\, dy$$

$$\vec{A} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$$

$$\vec{A}\cdot d\vec{s} = yz\, dx\, dy$$

$$\int_{S_1} \vec{A}\cdot d\vec{S} = \int_{S_1} [yz\, dx\, dy]$$

Along S_1 $z = 1$

$$= \int\int y\, dy\, dx = \int_0^1 y\, dy \int_0^1 dx = \tfrac{1}{2}$$

$$\int_{S_2} \vec{A}\cdot d\vec{S} \qquad d\vec{S} = -\hat{k}\, dx\, dy$$

$$\vec{A} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$$

$$\vec{A}\cdot d\vec{S} = -yz\, dx\, dy$$

$$\int_{S_2} \vec{A}\cdot d\vec{S} = -\int_{S_2} yz\, dx\, dy$$

Along S_2 $z = 0$

$$-\int_{S_2} yz\, dx\, dy = 0$$

$$\int_{S_3} \vec{A}\cdot d\vec{S} \qquad d\vec{S} = -\hat{j}\,dx\,dz$$

$$\vec{A} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$$

$$\vec{A}\cdot d\vec{S} = +y^2\,dx\,dz$$

$$\int_{S_3} \vec{A}\cdot d\vec{S} = \int_{S_3} y^2\,dx\,dz$$

Along S_3

$y = 0$

$dy = 0$

$$\int_{S_3} \vec{A}\cdot d\vec{S} = 0$$

$$\int_{S_4} \vec{A}\cdot d\vec{S} \qquad d\vec{S} = +\hat{j}\,dx\,dz$$

$$\vec{A} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$$

$$\vec{A}\cdot d\vec{S} = -y^2\,dx\,dz$$

$$\int_{S_4} \vec{A}\cdot d\vec{S} = -\int_{S_4} y^2\,dx\,dz$$

Along S_4

$y = 1$

$$\int_{S_4} \vec{A}\cdot d\vec{S} = -\int_0^1 dx \int_0^1 dz = -1$$

$$\int_{S_5} \vec{A}\cdot d\vec{S} \qquad dS = \hat{i}\, dy\, dz$$

$$\vec{A} = 4xz\,\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$\vec{A}\cdot d\vec{S} = 4xz\, dy\, dz$$

$$\int_{S_5} \vec{A}\cdot d\vec{S} = \int_{S_5} 4xz\, dy\, dz$$

Along S_5 $x = 1$

$$\int_{S_5} \vec{A}\cdot d\vec{S} = 4\iint z\, dz\, dy = 4\int_0^1 dy \int_0^1 z\, dz$$

$$\int_{S_5} \vec{A}\cdot d\vec{S} = 4\cdot 1\cdot \frac{1}{2} = 2$$

$$\int_{S_6} \vec{A}\cdot d\vec{S} \qquad dS = -\hat{i}\, dy\, dz$$

$$\vec{A} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$\vec{A}\cdot d\vec{S} = -4xz\, dy\, dz$$

$$\int_{S_6} \vec{A}\cdot d\vec{S} = -4\int_{S_6} xz\, dy\, dz$$

Along S_6, $x = 0$

$$\int_{S_6} \vec{A}\cdot d\vec{S} = 0$$

$$\oint_S \vec{A} \cdot d\vec{S} = \frac{1}{2} + 0 + 0 - 1 + 2 = \frac{3}{2}$$

Problem 6

Determine the flux of $\vec{G}$

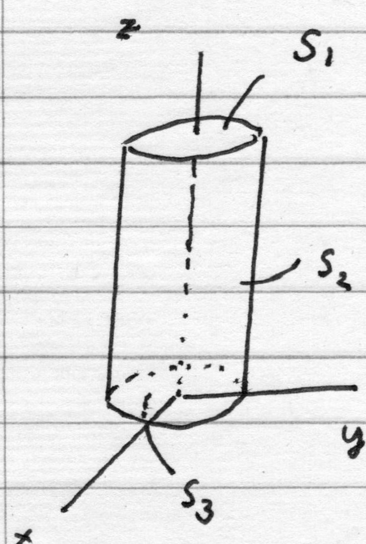
$$\oint_S \vec{G} \cdot d\vec{s}$$

where $\vec{G}(r) = 10e^{-2z}(\rho\hat{\rho} + \hat{k})$

Out of the entire surface of the cylinder

$\rho = 1$

$0 \leq z \leq 1$.



Let us start with S_1

$$\int_{S_1} \vec{G} \cdot d\vec{s}$$

$$\vec{G} = 10e^{-2z}(\rho\hat{\rho} + \hat{k})$$

$$d\vec{s} = \rho\, d\rho\, d\phi\, \hat{k}$$

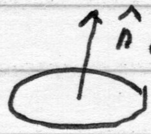
[Check Problem 1]

$$\vec{G} \cdot d\vec{s} = 10e^{-2z}\rho\, d\rho\, d\phi$$

$$\int_{S_1} \vec{G} \cdot d\vec{s} = \iint 10e^{-2z}\rho\, d\rho\, d\phi = \int_0^{2\pi}\int_0^1 10e^{-2}\rho\, d\rho\, d\phi$$

where $z = 1$

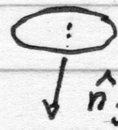
$$\int_{S_1} \vec{G}\cdot d\vec{S} = 10e^{-2}\,\frac{2\pi}{2} = 5e^{-2}\,2(\pi) = 10\pi e^{-2}$$



$\hat{n}_1$ $\qquad d\vec{S}_1 = \hat{n}_1\, dS_1$

For S_3

$$\vec{G}\cdot d\vec{S} = \left(10e^{-2z}[\rho\hat{\rho}+\hat{k}]\right)\cdot d\vec{S}$$



$\hat{n}_3$ $\qquad \vec{S}_3 = \hat{n}_3 S_3 = -\hat{k} S_3$

$$\int_{S_1} \vec{G}\cdot d\vec{S} = -\int_0^{2\pi}\!\!\int_0^{1} 10e^{-2z}\rho\, d\rho\, d\phi\Big|_{z=0} = -10e^{0}\,2\pi\,\frac{1}{2}$$

$$= -5e^{0}\,2\pi$$

$$= -10\pi$$

For S_2

$$\vec{G}\cdot d\vec{S} = \left(10e^{-2z}[\rho\hat{\rho}+\hat{k}]\right)\cdot d\vec{S}$$

For $\vec{S}_2 \Rightarrow d\vec{S}_2 = \hat{n}\, dS_2$ $\qquad dS = \rho\, d\phi\, dz\, \hat{\rho}$

[Check Problem ?]

$$\hat{n} = \hat{\rho}$$

$$\int_{S_3} \vec{G}\cdot d\vec{S} = \iint 10 e^{-2z} \rho^2 d\phi \, dz$$

Since $\rho = 1$

$$\int_{S_3} \vec{G}\cdot d\vec{S} = 10\int_0^{2\pi}\int_0^1 e^{-2z} \, dz \, d\phi$$

$$\int_{S_3} \vec{G}\cdot d\vec{S} = 10(2\pi)\int_0^1 e^{-2z} \, dz$$

$$u = -2z \qquad du = -2dz$$

$$\int_{S_3} \vec{G}\cdot d\vec{S} = \frac{10(2\pi)}{-2}\int_0^{-2} e^{u} \, du$$

$$= -\pi 10 e^{u}\Big|_0^{-2} = -10\pi(e^{-2} - 1)$$

$$= 10\pi(1 - e^{-2})$$

$$\oint_S \vec{G}\cdot d\vec{S} = -10\pi + 10\pi e^{-2} + 10\pi - 10\pi e^{-2}$$

$$\boxed{\oint_S \vec{G}\cdot d\vec{S} = 0}$$

Problem 7

Determine the flux of $\vec{D}$

$$\oint_S \vec{D} \cdot d\vec{S}$$

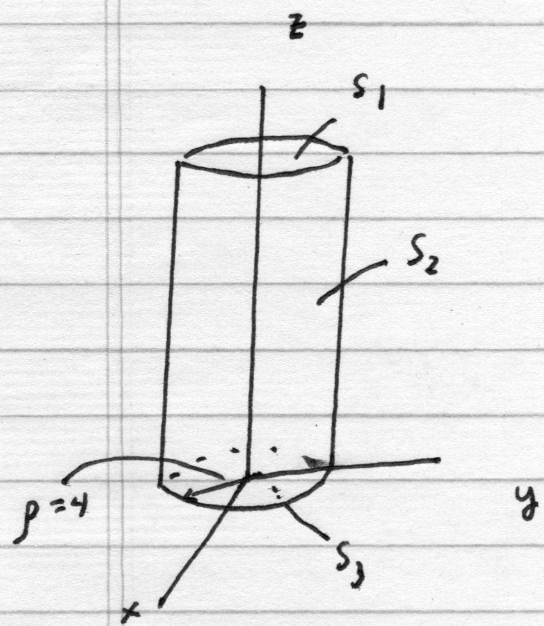
where

$$\vec{D} = \rho^2 \cos^2\phi \, \hat{\rho} + z \sin\phi \, \hat{\phi}$$

over the closed surface of the cylinder

$$\rho = 4$$

$$0 \leq z \leq 1 .$$



$$\oint_S \vec{D} \cdot d\vec{S} = \int_{S_1} \vec{D} \cdot d\vec{S} + \int_{S_2} \vec{D} \cdot d\vec{S} + \int_{S_3} \vec{D} \cdot d\vec{S}$$

For S_1 $\quad d\vec{S} = \rho \, d\rho \, d\phi \, \hat{k}$

From Problem 1

$$\vec{D} = \rho^2 \cos^2\phi \, \hat{\rho} + z \sin\phi \, \hat{\phi}$$

$$\boxed{\vec{D} \cdot d\vec{S} = 0}$$

For S_3 $\qquad \vec{ds} = -\rho\, d\rho\, d\phi\, \hat{k}$ [From Problem 1]

$$\vec{D} = \rho^2 \cos^2\phi\, \hat{\rho} + z \sin\phi\, \hat{\phi}$$

$$\boxed{\vec{D}\cdot\vec{ds} = 0}$$

For S_2 $\qquad \vec{ds} = \rho\, d\phi\, dz\, \hat{\rho}$

$$\vec{D} = \rho^2 \cos^2\phi\, \hat{\rho} + z \sin\phi\, \hat{\phi}$$

$$\vec{D}\cdot\vec{ds} = \rho^3 \cos^2\phi\, d\phi\, dz$$

$$\int_{S_2} \vec{D}\cdot\vec{ds} = \int_{S_2} \rho^3 \cos^2\phi\, d\phi\, dz$$

Note $\rho = 4$

$$\int_{S_2} \vec{D}\cdot\vec{ds} = 64 \int_0^1 dz \int_0^{2\pi} \cos^2\phi\, d\phi$$

$$\int_{S_2} \vec{D}\cdot\vec{ds} = 64 \int_0^{2\pi} \left[\frac{\cos(2\phi) + 1}{2} \right] d\phi$$

$$\int_{S_2} \vec{D}\cdot\vec{ds} = 64 \left[\int_0^{2\pi} \frac{\cos 2\phi}{2}\, d\phi + \int_0^{2\pi} \frac{d\phi}{2} \right]$$

$$\int_{S_2} \vec{D}\cdot d\vec{S} = 64\left[\pi + \frac{1}{2}\int_0^{2\pi} \cos 2\phi \, d\phi\right]$$

$$u = 2\phi \qquad du = 2\,d\phi$$

ϕ	u
0	0
2π	4π

$$\int_{S_2} \vec{D}\cdot d\vec{S} = 64\pi + \frac{32}{2}\int_0^{4\pi} \cos u \, du$$

$$\int_{S_2} \vec{D}\cdot d\vec{S} = 64\pi + \cancelto{0}{16 \sin u \Big|_0^{4\pi}}$$

$$\int_{S_2} \vec{D}\cdot d\vec{S} = 64\pi$$

$$\oint_S \vec{D}\cdot d\vec{S} = 0 + 0 + 64\pi = 64\pi$$

Problem 8

If

$$g(\theta,\phi) = r^2$$

evaluate

$$\int g(\theta,\phi)\, dV$$

over a hemisphere of radius 1, $z \geq 0$ centered at the origin.

$$\int g(\theta,\phi) = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1} r^2\, r^2 dr \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{r^5}{5}\Big|_0^1 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{5}\left[\int_0^{2\pi} (-\cos\theta)\Big|_0^{\frac{\pi}{2}}\right] d\phi$$

$$\frac{1}{5}\, 2\pi \quad = 1.2566$$

Problem 9

Evaluate the line integral

$$\oint_C \vec{A} \cdot d\vec{r}$$

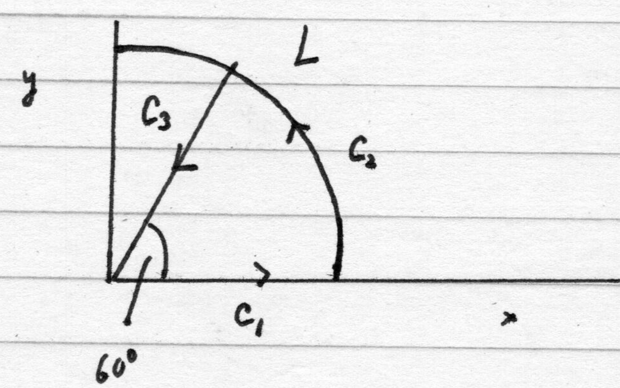
where

$$\vec{A} = \hat{\rho}\, \rho \cos\phi + \hat{k}\, z \sin\phi$$

and C is the edge L of the wedge defined by,

$$0 \le \rho \le 2$$
$$0 \le \phi \le 60^\circ$$
$$z = 0$$

as shown in the figure below



$$C = C_1 + C_2 + C_3$$

$$\oint_C \vec{A}\cdot d\vec{r} = \int_{C_1} \vec{A}\cdot d\vec{r} + \int_{C_2} \vec{A}\cdot d\vec{r} + \int_{C_3} \vec{A}\cdot d\vec{r}$$

$$\int_{C_1} \vec{A}\cdot d\vec{r} \qquad d\vec{r} = \hat{\rho}\, d\rho + \hat{\phi}\, \rho\, d\phi + \hat{k}\, dz$$

(Cf. Problem 4-1)

$$\vec{A} = \hat{\rho}\,\rho\cos\phi + \hat{k}\, z\sin\phi$$

Along C_1, $z=0$, $\phi=0$

$$\vec{A}\cdot d\vec{r} = \rho\, d\rho\cos\phi + 0$$

since $\hat{\phi} \perp \hat{\rho}$

$$\int_{C_1} \vec{A}\cdot d\vec{r} = \int \rho\, d\rho\cos\phi = \int_0^2 \rho\, d\rho = 2$$

$$\int_{C_2} \vec{A}\cdot d\vec{r}$$

Along C_2 $z=0$

$$\vec{A}\cdot d\vec{r} = \rho\, d\rho\cos\phi + 0$$

since $\hat{\phi} \perp \hat{\rho}$ again

$$\int_{C_2} \rho\, d\rho\cos\phi \to 0$$

Along C_2 $\rho =$ constant

$d\rho = 0$

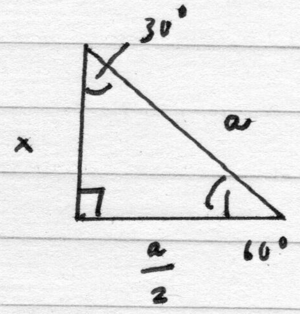
$\int_{C_3} \vec{A}\cdot d\vec{r}$ Along C_3 $z=0$

$$\vec{A}\cdot d\vec{r} = \rho\, d\rho \cos\phi$$

Along C_3 $\phi = 60^\circ$

$$\vec{A}\cdot d\vec{r} = \rho\, d\rho \cos 60^\circ$$

Recall some useful trigonometry



(Not drawn to scale)

$$x^2 + \frac{a^2}{4} = a^2$$

$$x^2 = \frac{3a^2}{4}$$

$$x = \sqrt{3}\,\frac{a}{2}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 30^\circ = \frac{1}{2}$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\vec{A}\cdot d\vec{r} = \rho\, d\rho \cos 60^\circ = \frac{1}{2}\rho\, d\rho$$

$$\int_{C_3} \vec{A}\cdot d\vec{r} = \frac{1}{2}\int_2^0 \rho\, d\rho = \frac{1}{2}\,\frac{\rho^2}{2}\Big|_2^0 = -\frac{1}{2}\,\frac{4}{2} = -1$$

$$\oint_C \vec{A}\cdot d\vec{r} = \overbrace{\int_{C_1} \vec{A}\cdot d\vec{r}}^{2} + \overbrace{\int_{C_2} \vec{A}\cdot d\vec{r}}^{0} + \overbrace{\int_{C_3} \vec{A}\cdot \vec{r}}^{-1}$$

$$\boxed{\oint_C \vec{A}\cdot d\vec{r} = 1}$$