USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET II
(due Friday, July 3, 2020)

Problem 1

Evaluate the following integral by using trigonometric substitution. In this problem a is a constant and we ignore the constant of integration.

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Problem 2

Evaluate the following integral by using trigonometric substitution. In this problem a is a constant and we ignore the constant of integration.

$$\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2 + x^2}}$$

Problem 3

This integral is quite involved to evaluate but it uses a very clever trick. In this problem a is a constant and we ignore all constants of integration. First show that the following is true using trigonometric substitution

$$\int \frac{dx}{(a^2 + x^2)^{\frac{1}{2}}} = \int \sec\theta \, d\theta$$

Next multiply both the numerator and denominator of this result by the term $\sec\theta + \tan\theta$. Expand this result out and then next by looking at the relationship between the denominator and numerator use integration by substitution to get the desired result. In this problem a is a constant and we ignore the constant of integration.

$$\int \frac{dx}{(a^2 + x^2)^{\frac{1}{2}}} = \ln\left[\frac{x}{a} + \frac{(a^2 + x^2)^{\frac{1}{2}}}{a}\right]$$

Problem 4

Evaluate the following integral by first using the method of partial fractions and then using integration by substitution. In this problem both a and b are constants and we ignore the constant of integration.

$$\int \frac{dx}{x\,(ax + b)} = \frac{1}{b}\ln\left(\frac{x}{ax + b}\right)$$

Problem 5

Evaluate the following integral by first using the method of partial fractions and then using integration by substitution. In this problem a is a constant.

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left(\frac{x - a}{x + a}\right)$$

Problem 6

Evaluate the following integral with no hints

$$\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}}$$

Problem 7

Evaluate the following integral with no hints

$$\int \frac{x\,dx}{(ax + b)^2}$$

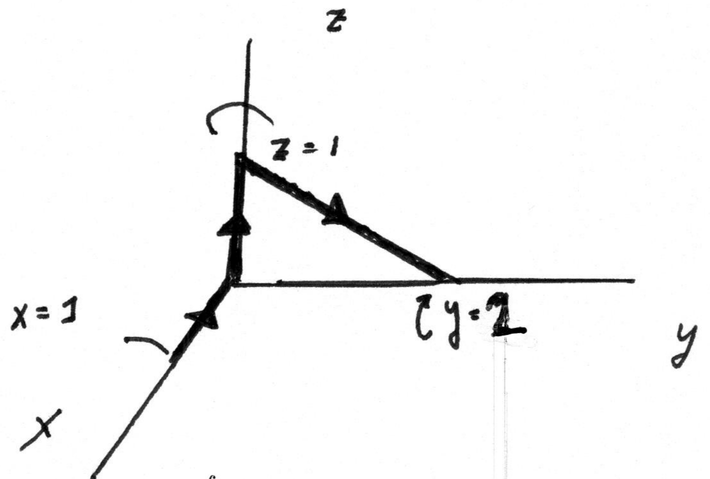
Problem 8

Evaluate the following integral with no hints

$$\int \frac{x^2\, dx}{(ax+b)^2}$$

Problem 9

Evaluate the following line integral of $\vec{H}$ around the closed path C in the figure below



$$\int_C \vec{H} \cdot d\vec{r}$$

where

$$\vec{H} = (x-y)\hat{\imath} + (x^2 + zy)\hat{\jmath} + 5yz\hat{k}$$

Problem 10

We will discuss later in this course that the line intergral below is regarded as the work done in moving a particle from a point $\vec{A}$ to a point $\vec{B}$ in space

$$\int_A^B \vec{F} \cdot d\vec{r}$$

For the following choice of $\vec{F}$

$$\vec{F} = (2xy - 3)\hat{\imath} + x^2\hat{\jmath}$$

find the work done by $\vec{F}$ along each of the paths indicated below from the point (1, 0) to (0,1):

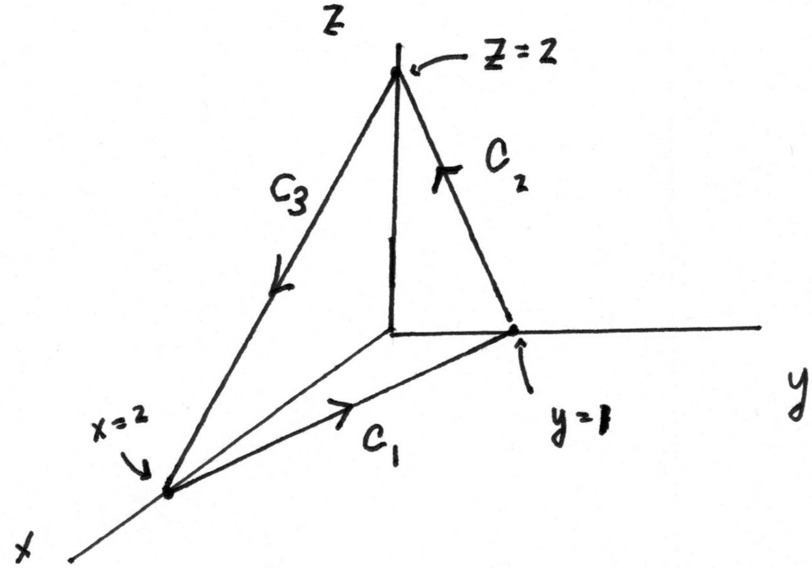
(i) Path 1: along the straight line from (1, 0) to (0, 1)

(i) Path 2: along the circular arc from (1, 0) to (0, 1)

(iii) Path 3: first along the straight line from (1, 0) to (1, 1) and then along the straight line from (1,1) to (0,1)

Problem 11

Evaluate the following line integral of $\vec{M}$ around the closed path C in the figure below where C is the part of the plane x + 2y + z =2 in the first octant



$$\oint_C \vec{M} \cdot d\vec{r}$$

and

$$\vec{M} = (y - x)\hat{\imath} + (x - z)\hat{\jmath} + (x - y)\hat{k}$$

Problem 12

Evaluate the following line integral below

$$\int_C \frac{-y\,\hat{\imath} + x\,\hat{\jmath}}{\sqrt{x^2+y^2}} \cdot d\vec{r}$$

where C is a circle of radius a in the xy-plane, with the center at the origin, traced out counterclockwise.

Problem 1

Integrate $\int \frac{dx}{x^2+a^2}$

Let $[x^2+a^2] = a^2\left[1+\left(\frac{x}{a}\right)^2\right]$

Let $\tan\theta = \frac{x}{a}$ or $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$ [Since a is a constant

$$\int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2[1+\tan^2\theta]}$$

$$= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta = \frac{1}{a}\int d\theta = \frac{\theta}{a}$$

Finally since $\tan\theta = \frac{x}{a}$

$\theta = \tan^{-1}\left(\frac{x}{a}\right)$

and

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$$

Problem 2

Integrate

$$\int \frac{dx}{[a^2+x^2]^{3/2}}$$

$$a^3\left[1+\left(\frac{x}{a}\right)^2\right]^{3/2}$$

Let $\quad \tan\theta = \dfrac{x}{a} \qquad x = a\tan\theta$, where a is a constant

$$dx = a\sec^2\theta\, d\theta$$

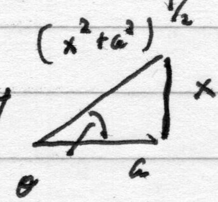
$$\left[1+\left(\frac{x}{a}\right)^2\right]^{3/2} = \left[1+\tan^2\theta\right]^{3/2} = \left[\sec^2\theta\right]^{3/2}$$

$$= \sec^3\theta$$

$$\int \frac{dx}{[a^2+x^2]^{3/2}} = \int \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int \cos\theta\, d\theta$$

$$\int \frac{dx}{[a^2+x^2]^{3/2}} = \frac{1}{a^2}\sin\theta$$

Using this right triangle to relate θ to a, x



$$\boxed{\int \frac{dx}{[a^2+x^2]^{3/2}} = \frac{1}{a^2}\,\frac{x}{[x^2+a^2]^{1/2}}}$$

where a is a constant

Problem 3

Integrate $\int \frac{dx}{[a^2+x^2]^{1/2}}$

Let

$$\int \frac{dx}{[a^2+x^2]^{1/2}} = \int \frac{dx}{a\left[1+\left(\frac{x}{a}\right)^2\right]^{1/2}}$$

and let $\tan\theta = \frac{x}{a}$

$$dx = a\sec^2\theta\, d\theta$$

$$\left[1+\left(\frac{x}{a}\right)^2\right]^{1/2} = \left[1+\tan^2\theta\right]^{1/2} = \sec\theta$$

$$\int \frac{dx}{[a^2+x^2]^{1/2}} = \int \frac{a\ \sec^2\theta\, d\theta}{a\ \sec\theta} = \int \sec\theta\, d\theta$$

$$\int \frac{\sec\theta\,[\tan\theta+\sec\theta]\,d\theta}{\sec\theta+\tan\theta} = \int \frac{[\sec^2\theta + \tan\theta\sec\theta]\,d\theta}{\sec\theta+\tan\theta}$$

Let $u = \tan\theta + \sec\theta$

$$du = \sec^2\theta\, d\theta + \frac{0-(-\sin\theta)}{\cos^2\theta}\, d\theta$$

$$du = [\sec^2\theta + \tan\theta\sec\theta]\, d\theta$$

Thus

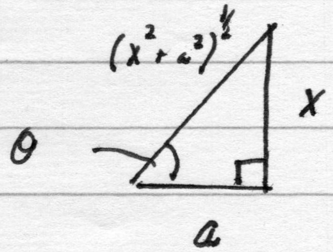
$$\int \frac{dx}{[a^2+x^2]^{1/2}} = \int \frac{du}{u} = \ln u = \ln[\tan\theta + \sec\theta]$$

where $\quad u = \tan\theta + \sec\theta$

$$du = [\sec^2\theta + \tan\theta\sec\theta]d\theta$$

Now let us recall

$$\tan\theta = \frac{x}{a}$$



so from the diagram

$$\cos\theta = \frac{a}{[x^2+a^2]^{1/2}}$$

or $\quad \sec\theta = \dfrac{[x^2+a^2]^{1/2}}{a}$

$$\int \frac{dx}{[a^2+x^2]^{1/2}} = \ln\left[\left(\frac{x}{a}\right) + \left(\frac{[x^2+a^2]^{1/2}}{a}\right)\right]$$

$$= \ln\left[\frac{x}{a} + \frac{(x^2+a^2)^{1/2}}{a}\right]$$

Problem 4

Evaluate the integral

$$\int \frac{dx}{x(ax+b)}$$

Use the method of partial fractions

$$\frac{1}{x(ax+b)} = \frac{C_1}{x} + \frac{C_2}{(ax+b)} = \frac{1}{x(ax+b)}$$

$$(ax+b)C_1 + C_2 x = 1$$

$$(C_1 a + C_2)x + C_1 b = 1$$

$$C_1 a + C_2 \Rightarrow 0$$
$$C_1 b = 1$$

$$C_1 a + C_2 = 0$$
$$C_1 b = 1$$

$$\begin{cases} C_1 ab + C_2 b = 0 & (1) \\ C_1 ab = a & (2) \end{cases}$$

Subtract (1) - (2)

$$C_2 b = -a \qquad C_2 = -\frac{a}{b}$$

$$C_1 = \frac{1}{b}$$

$$\int \frac{dx}{x(ax+b)} = \int \frac{dx}{bx} - \frac{a}{b}\int \frac{dx}{(ax+b)}$$

$$= \frac{1}{b}\int \frac{dx}{x} - \frac{a}{b}\int \frac{dx}{(ax+b)}$$

Now let us do these integrals by substitution

$$\frac{1}{b}\int \frac{dx}{x} \qquad \text{Let } u = x \text{ , } du = dx$$

$$\frac{1}{b}\int \frac{du}{u} = \frac{1}{b}\ln u = \frac{1}{b}\ln x$$

$$-\frac{a}{b}\int \frac{dx}{(ax+b)} \qquad \text{Let } u = ax+b, \ du = a\,dx$$

$$-\frac{a}{b}\int \frac{dx}{(ax+b)} = -\frac{a}{b}\,\frac{1}{a}\int \frac{du}{u} = -\frac{1}{b}\ln u$$

$$= -\frac{1}{b}\ln(ax+b)$$

$$\int \frac{dx}{x[ax+b]} = \frac{1}{b}\ln x - \frac{1}{b}\ln(ax+b)$$

$$\int \frac{dx}{x[ax+b]} = \frac{1}{b}\ln\left(\frac{x}{ax+b}\right)$$

Problem 5

Integrate

$$\int \frac{dx}{x^2-a^2} \quad \text{where } a \text{ is a constant}$$

Use method of partial fractions

$$\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{C_1}{(x+a)} + \frac{C_2}{(x-a)} = \frac{1}{(x+a)(x-a)}$$

Since $\quad (x-a)C_1 + C_2(x+a) = 1$

$$xC_1 - aC_1 + C_2x + C_2a = 1$$

$$x(C_1+C_2) + a(C_2-C_1) = 1$$

or

$$C_1 + C_2 = 0$$
$$aC_2 - aC_1 = 1$$

which can be solved to yield

$$\begin{aligned} aC_1 + aC_2 &= 0 \\ aC_2 - aC_1 &= 1 \end{aligned} \Rightarrow 2aC_2 = 1$$

or $\quad C_2 = \frac{1}{2a}$

$\therefore \quad C_1 = -\frac{1}{2a}$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\left[-\int \frac{dx}{(x+a)} + \int \frac{dx}{(x-a)}\right]$$

Let $u = x+a$ $\quad du = dx$ $\quad$ and $\quad v = x-a$ $\quad dv = dx$
(1st integral)

and integrate the result to give

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\left[-\ln(x+a)\right] + \frac{1}{2a}\left[\ln(x-a)\right]$$

or

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right)$$

Problem 6

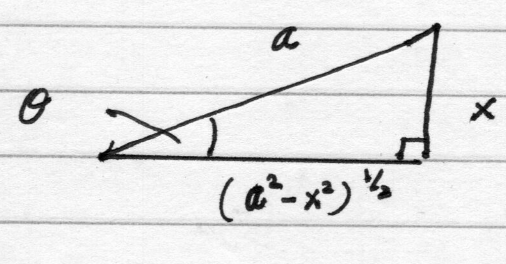
Integrate

$$\int \frac{dx}{\sqrt{a^2-x^2}}$$

Clearly we want to take the positive square root so

$$a > x$$
$$a^2 > x^2$$
$$a^2 - x^2 > x^2 - x^2 = 0$$
$$(a^2 - x^2)^{1/2} > 0$$

We can use trigonometric substitution and define the angle θ such that



and

$$\sin\theta = \frac{x}{a}$$

$$\sqrt{a^2-x^2} \rightarrow a\left(1-\left(\frac{x}{a}\right)^2\right)^{1/2} = a\left[1-\sin^2\theta\right]^{1/2}$$

$$\sqrt{a^2 - x^2} = a \cos \theta$$

$$dx = a \cos \theta \, d\theta$$

$$\boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos \theta \, d\theta}{a \cos \theta} = \int d\theta = \theta = \sin^{-1}\left(\frac{x}{a}\right)}$$

Problem 7-

Evaluate the integral

$$\int \frac{x\,dx}{[ax+b]^2}$$

Let $\quad U = ax + b$

Therefor $\quad ax = U - b$

$$x = \frac{U}{a} - \frac{b}{a}$$

$$dx = \frac{1}{a}\,dU$$

$$[ax+b]^2 = U^2$$

$$\int \frac{x\,dx}{[ax+b]^2} = \frac{1}{a}\int \left[\frac{U}{a} - \frac{b}{a}\right] \frac{dU}{U^2}$$

$$\int \frac{x\,dx}{[ax+b]^2} = \frac{1}{a^2}\int \frac{dU}{U} - \frac{b}{a^2}\, U^{-2}\,dU$$

$$\int \frac{x\,dx}{[ax+b]^2} = \frac{1}{a^2}\ln U - \frac{b}{a^2}\,\frac{U^{-1}}{(-1)}$$

$$\int \frac{x\,dx}{[ax+b]^2} = \frac{1}{a^2}\ln[ax+b] + \frac{b}{a^2}\,\frac{1}{[ax+b]}$$

Problem 8

Evaluate the following integral

$$\int \frac{x^2\,dx}{[ax+b]^2}$$

Let $u = ax + b$

$ax = u - b$

$x = \frac{u-b}{a}$

$[ax+b]^2 = u^2$

$dx = \frac{1}{a}\,du$

$$\int \frac{x^2 dx}{[ax+b]^2} = \frac{1}{a^2}\int \frac{[u-b]^2\,du}{a\;u^2}$$

$$\int \frac{x^2 dx}{[ax+b]^2} = \frac{1}{a^3}\int\left[\frac{u^2 - 2ub + b^2}{u^2}\right]du$$

$$\int \frac{x^2 dx}{[ax+b]^2} = \frac{1}{a^3}\left[\int du \quad -2b\int\frac{du}{u} \quad +\int b^2 u^{-2}\,du\right]$$

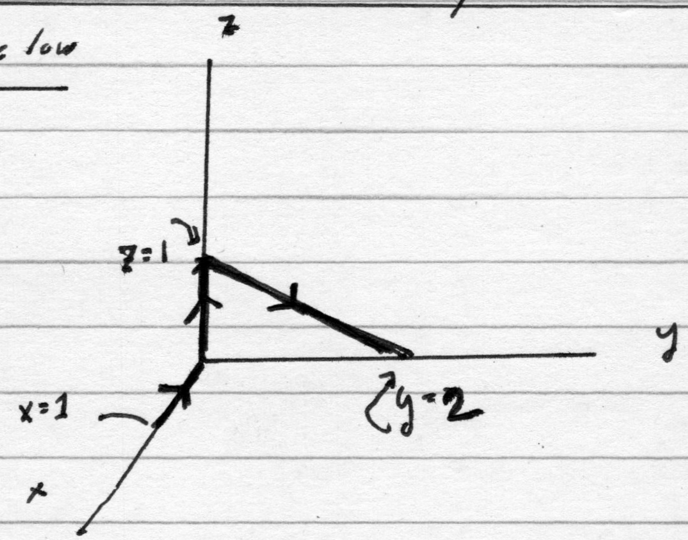
$$\int \frac{x^2 dx}{[ax+b]^2} = \frac{u}{a^3} \quad -\frac{2b}{a^3}\ln u \quad + \frac{b^2}{a^3}\,\frac{u^{-1}}{(-1)}$$

$$\int \frac{x^2 dx}{[ax+b]^2} = \frac{ax+b}{a^3} - \frac{2b}{a^3} \ln[ax+b] - \frac{b^2}{a^3} \frac{1}{[ax+b]}$$

$$\int \frac{x^2 dx}{[ax+b]^2} = \frac{ax+b}{a^3} - \frac{2b}{a^3} \ln[ax+b] - \frac{b^2}{a^3} \frac{1}{[ax+b]}$$

Problem 9

Evaluate the following line integral of $\vec{H}$ around the closed path C in the figure below



$$\int_C \vec{H}\cdot d\vec{r}$$

Where

$$\vec{H} = (x-y)\hat{i} + (x^2+zy)\hat{j} + 5yz\hat{k}$$

$$d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$\vec{H}\cdot d\vec{r} = (x-y)\,dx + (x^2+zy)\,dy + 5yz\,dz$$

$$\int_C \vec{H}\cdot d\vec{r} = \int_C (x-y)dx + (x^2+zy)dy + 5yz\,dz$$

There are three paths in C

Path ①: x is a variable

$z = 0 \qquad dz = 0$

$y = 0 \qquad dy = 0$

$$\int_{C_1} (0 + x)\,dx + 0 + 0 = \int_1^0 x\,dx =$$

$$\frac{x^2}{2}\Big|_1^0 = -\frac{1}{2}$$

Path ②: z is a variable

$x = 0 = y = 0$

$dx = dy = 0$

$$\int_{C_2} 0 + 0 + 0 = 0$$

Path ③ A line in yz-plane

$$z = -\frac{1}{2}y + 1 = 1 - \frac{y}{2}$$

N.B. When $y = 0$, $z = 1$

When $y = 2$, $z = 0$

$x = 0 = dx = 0$

$$\int_C \vec{H}\cdot d\vec{r} = \overset{-1/2}{\cancel{\int_{C_1} \vec{H}\cdot d\vec{r}}} + \overset{0}{\cancel{\int_{C_2} \vec{H}\cdot d\vec{r}}} + \int_{C_3} \vec{H}\cdot d\vec{r}$$

$$\int_{C_3} \vec{H}\cdot d\vec{r} = \int_{C_3} \left[(x-y)\,dx + (x^2+zy)\,dy + 5yz\,dz \right]$$

$$\int_{C_3} \vec{H}\cdot d\vec{r} = \int \left[(0) + (0+zy)\,dy + 5yz\,dz \right]$$

$$= \int zy\,dy + \int 5yz\,dz$$

Let us choose to express every thing in terms of y

$$z = 1 - \frac{y}{2}$$

$$dz = -\frac{1}{2}\,dy$$

$$\int_{C_3} \vec{H}\cdot d\vec{r} = \int_{y=0}^{y=2} \left[1-\frac{y}{2}\right] y\,dy + 5\int_{y=0}^{y=2} y\left[1-\frac{y}{2}\right]dy\left(-\frac{1}{2}\right)$$

$$\int_{C_3} \vec{H}\cdot d\vec{r} = \int_0^2 \left[y - \frac{y^2}{2}\right] dy - \frac{5}{2}\int_0^2 \left[y - \frac{y^2}{2}\right] dy$$

$$\int_{C_3} \vec{H}\cdot d\vec{r} = \left[\frac{y^2}{2} - \frac{y^3}{6}\right]_0^2 - \frac{5}{2}\left[\frac{y^2}{2} - \frac{y^3}{6}\right]_0^2$$

$$\int_{C_3} \vec{H}\cdot d\vec{r} = \frac{4}{2} - \frac{8}{6} - \frac{5}{2}\left[\frac{4}{2} - \frac{8}{6}\right]$$

$$\int_{C_3} \vec{H}\cdot d\vec{r} = \frac{4}{2} - \frac{8}{6} - \frac{20}{4} + \frac{40}{12}$$

$$\int_{C_3} \vec{H}\cdot d\vec{r} = \frac{4}{2} - \frac{10}{2} - \frac{8}{6} + \frac{20}{6}$$

$$\int_{C_3} \vec{H}\cdot d\vec{r} = -\frac{6}{2} + \frac{12}{6} = -3 + 2 = -1$$

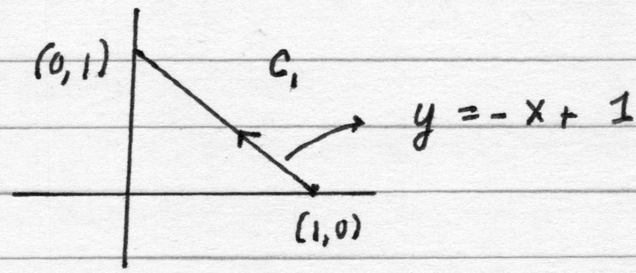
In summary,

$$\boxed{\int_C \vec{H}\cdot d\vec{r} = -\frac{1}{2} + 0 - 1 = -\frac{3}{2}}$$

Problem 10

$$\vec{F} = (2xy - 3)\hat{i} + x^2\hat{j}$$

(i) Path 1



$$W = \int_{C_1} \vec{F}\cdot d\vec{r} = \int (F_x\hat{i} + F_y\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$W = \int_{C_1} F_x\,dx + F_y\,dy$$

$$W = \int_{C_1} (2xy - 3)\,dx + x^2\,dy$$

Along C_1 $\quad y = -x + 1$

$\qquad\qquad\quad dy = -dx$

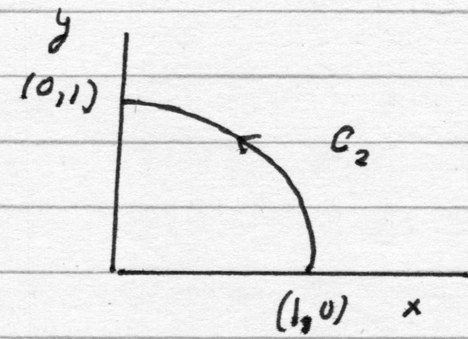
$$W = \int_1^0 [2x(1-x) - 3]\,dx - \int_1^0 x^2\,dx$$

$$W = \int_1^0 [2x - 2x^2 - 3]\,dx - \int_1^0 x^2\,dx$$

$$W = \left[\frac{2x^2}{2} - \frac{2}{3}x^3 - 3x\right]_1^0 - \frac{x^3}{3}\bigg|_1^0$$

$$W = -1 + \frac{2}{3} + 3 + \frac{1}{3} = 1 + 3 - 1 = 3$$

(iii) Path 2



$$C_2 \Rightarrow x^2 + y^2 = 1$$

$$W = \int_{C_2} [2xy - 3]\, dx + \int_{C_2} x^2\, dy$$

Change to plane polar coordinates

$$\left.\begin{aligned} x &= r\cos\theta = \cos\theta \\ y &= r\sin\theta \quad \sin\theta \end{aligned}\right\} \text{ since } r = 1$$

$$dx = -\sin\theta\, d\theta$$
$$dy = \cos\theta\, d\theta$$

$$W = \int_{C_2} \left[2\cos\theta \sin\theta - 3 \right] (-\sin\theta \, d\theta)$$

$$+ \int_{C_2} \cos^2\theta \left[\cos\theta \, d\theta \right]$$

$$W = \int_0^{\pi/2} \left[-2\cos\theta \sin^2\theta \, d\theta + 3\sin\theta \, d\theta \right]$$

$$+ \int_0^{\pi/2} \cos^3\theta \, d\theta$$

$$W = \int_0^{\pi/2} \left[-\cos\theta \left(+2\sin^2\theta - \cos^2\theta \right) + 3\sin\theta \right] d\theta$$

$$W = \int_0^{\pi/2} \left[-\cos\theta \left[2\sin^2\theta - [1 - \sin^2\theta] \right] + 3\sin\theta \right] d\theta$$

$$W = \int_0^{\pi/2} \left[-\cos\theta \left[3\sin^2\theta - 1 \right] + 3\sin\theta \right] d\theta$$

$$W = \underset{\text{I}}{-3\int_0^{\pi/2} \sin^2\theta \cos\theta \, d\theta} + \underset{\text{II}}{\int_0^{\pi/2} \cos\theta \, d\theta} + \underset{\text{III}}{3\int_0^{\pi/2} \sin\theta \, d\theta}$$

The last two integrals are easy

(II) $$\int_0^{\pi/2} \cos\theta \, d\theta = \sin\theta \Big|_0^{\frac{\pi}{2}} = 1$$

(III) $$3\int_0^{\pi/2} \sin\theta \, d\theta \quad -3\cos\theta \Big|_0^{\frac{\pi}{2}} \quad +3$$

while the first integral can be done by substitution

(I) $$-3\int_0^{\pi/2} \sin^2\theta \cos\theta \, d\theta$$

$$u = \sin\theta$$
$$du = \cos\theta \, d\theta$$
$$u^2 = \sin^2\theta$$

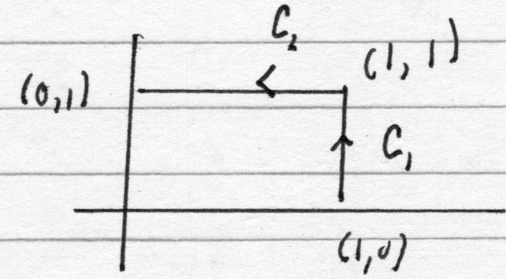
(I) $$-3\int u^2 du = \frac{-3u^3}{3} = -\left[\sin^3\theta\right]_0^{\frac{\pi}{2}}$$

$$= -\left[1-0\right] = -1$$

So the final result (I) + (II) + (III) is

$$W = \int_{C_2} [2xy-3]\, dx + \int_{C_2} x^2 dy = -1 + 1 + 3 = 3$$

(iii) Path 3



$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (2xy-3)\,dx + x^2\,dy$$

Along C_1 $x = 1$
$dx = 0$
y is a variable

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 dy = 1$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} (2xy-3)\,dx + x^2\,dy$$

Along C_2 $y = 1$
$dy = 0$
x is a variable

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_1^0 [2x-3]\,dx = \Big[x^2 - 3x\Big]_1^0 = -1+3 = 2$$

Along Path (iii) $C_1 + C_2 \Rightarrow$ result is $2+1 = 3$

Note that

$$\int_{\text{Path 1}} \vec{F}\cdot d\vec{r} = \int_{\text{Path 2}} \vec{F}\cdot d\vec{r} = \int_{\text{Path 3}} \vec{F}\cdot d\vec{r}$$

So that this vector $\vec{F}$ is very special as its result is path independent from $(1,0)$ to $(0,1)$

Problem 11

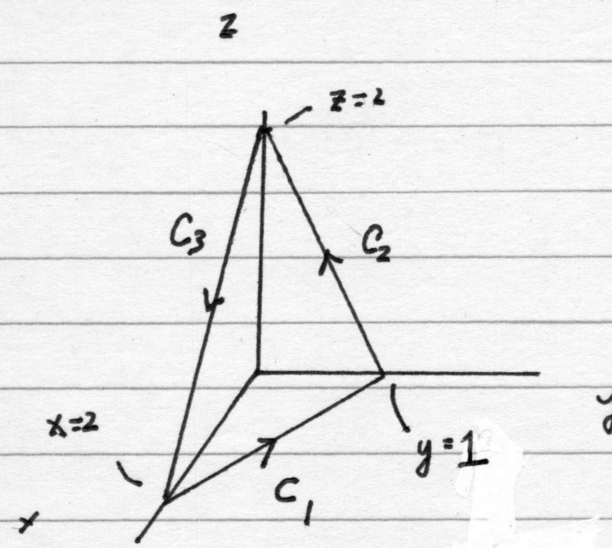
Suppose you are given the vector $\vec{M}$ which is defined as

$$\vec{M} = (y-x)\hat{i} + (x-z)\hat{j} + (x-y)\hat{k}$$

(i) Evaluate the line integral

$$\oint_C \vec{M} \cdot d\vec{r}$$

where C is the closed path defined in the figure below and $C = C_1 + C_2 + C_3$



and this plane above in the figure is defined by the surface.

$$x + 2y + z = 2$$

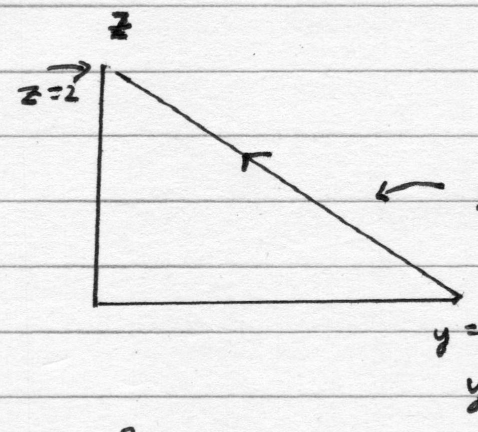
$$\int_{C_1} \vec{M}\cdot d\vec{r} = \int_{y=0}^{y=1} (y - 2 + 2y)(-2\,dy)$$

$$+ \int_{y=0}^{y=1} (2 - 2y)\,dy$$

$$\int_{C_1} \vec{M}\cdot d\vec{r} = \left[-2\frac{y^2}{2} + 4y - \frac{4y^2}{2}\right]_0^1 + 2y\Big|_0^1 - \frac{2y^2}{2}\Big|_0^1$$

$$\int_{C_1} \vec{M}\cdot d\vec{r} = -1 + 4 - 2 + 2 - 1 = 2$$

What about C_2?



in yz-plane
$x = 0$

$$2y + z = 2 \longrightarrow \boxed{y = -\frac{1}{2}z + 1}$$

or

$$z = 2 - 2y$$

$$x = 0,\ dx = 0$$

$$dz = -2\,dy \quad \text{or} \quad dy = \frac{-1}{2}\,dz$$

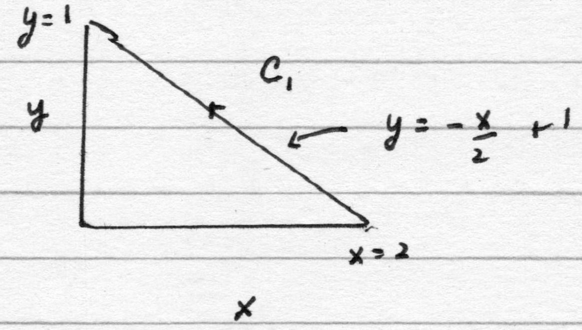
Let us first review C_1 where $z = 0$

Thus $x + 2y + z = 2$ becomes $z = 0$

$$x + 2y = 2$$

$$2y = 2 - x$$

$$y = \frac{-x}{2} + 1$$



$\boxed{\text{in } xy\text{-plane}}$

$$\vec{M} = (y-x)\,\hat{i} + (x-z)\,\hat{j} + (x-y)\,\hat{k}$$

$$d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$\vec{M}\cdot d\vec{r} = (y-x)\,dx + (x-z)\,dy + (x-y)\,dz$$

$$\int_{C_1} (\vec{M}\cdot d\vec{r}) = \int_{C_1} \vec{M}\cdot d\vec{r} = \int (y-x)\,dx + \int (x-z)\,dy + \int (x-y)\,dz$$

To evaluate this over y we realize and note

$$x = 2 - 2y \qquad z = 0$$

$$dx = -2\,dy \qquad dz = 0$$

$$\int_{C_2} \vec{H}\cdot d\vec{l} = \int (y-x)\,dx + \int (x-z)\,dy + \int (x-y)\,dz$$

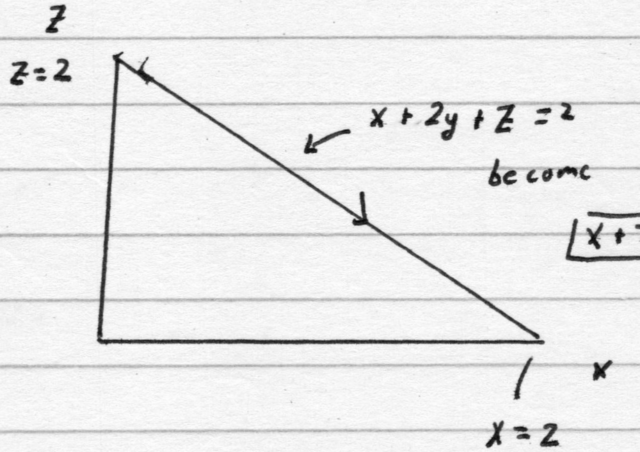
$$= \int -z\,dy + \int (x-y)\,dz$$

$$= \int_0^2 -z\left(-\frac{1}{2}\,dz\right) + \int_0^2 -\left[1-\frac{z}{2}\right]dz$$

$$+\frac{1}{2}\frac{z^2}{2}\Big|_0^2 + (-z)\Big|_0^2 + \frac{z^2}{4}\Big|_0^2$$

$$\frac{4}{4} - 2 + \frac{4}{4} = 0$$

Finally, the case for C_3!



XZ-plane

$y=0$

$y=0;\ dy=0;\qquad x=2-z;\qquad dx=-dz$

$dz=-dx$

$$\int_{C_3} \vec{H}\cdot d\vec{l} = \int (y-x)\,dx + \int (x-z)\,dy + \int (x-y)\,dz$$

$$\Big\downarrow \text{ becomes}$$

$$\int_{C_3} \vec{H}\cdot d\vec{l} = \int_0^2 -x\,dx + \int_0^2 x\,(-dx)$$

$$- \left.\frac{+x^2}{2}\right|_0^2 \qquad \left.-\frac{x^2}{2}\right|_0^2$$

$$2 \qquad -2 \qquad = -4$$

In summary

$$\oint_C \vec{M}\cdot d\vec{r} = \cancelto{2}{\int_{C_1} \vec{M}\cdot d\vec{r}} + \cancelto{0}{\int_{C_2} \vec{M}\cdot d\vec{r}} + \cancelto{-4}{\int_{C_3} \vec{M}\cdot d\vec{r}}$$

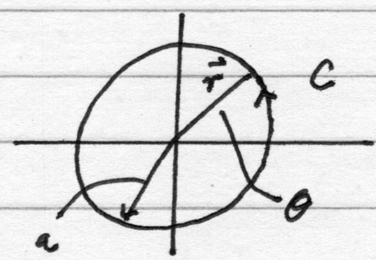
$$\boxed{\oint_C \vec{M}\cdot d\vec{r} = -2}$$

Problem 12

Evaluate the following line integral below

$$\int_C \left(\frac{-y\hat{\imath} + x\hat{\jmath}}{[x^2+y^2]^{1/2}}\right)\cdot d\vec{r}$$

Where C is a circle of radius a in the xy-plane, with the center at the origin, traced out counterclockwise.



$\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z = \hat{\imath}x + \hat{\jmath}y$ (Since $z = 0$ as we are in xy-plane)

$d\vec{r} = \hat{\imath}\,dx + \hat{\jmath}\,dy + \hat{k}\,dz$

$x = r\cos\theta \qquad dx = -r\sin\theta\, d\theta$

$y = r\sin\theta \qquad dy = r\cos\theta\, d\theta$, where r is a constant

$x^2 + y^2 = r^2[\cos^2\theta + \sin^2\theta] = r^2$

$(-y\hat{\imath} + x\hat{\jmath})\cdot d\vec{r} = (-y\hat{\imath} + x\hat{\jmath})\cdot(\hat{\imath}\,dx + \hat{\jmath}\,dy) =$

$-y\,dx + x\,dy$

$$\int_C \frac{[-y\hat{i} + x\hat{j}] \cdot d\vec{r}}{[x^2+y^2]^{1/2}} = \int_C \frac{-y\,dx + x\,dy}{[x^2+y^2]^{1/2}}$$

Let us do a change in variables to plane polar coordinates

$$\int_C \frac{-y\,dx + x\,dy}{[x^2+y^2]^{1/2}} = \int \frac{-r\sin\theta\,(-r\sin\theta)\,d\theta + r\cos\theta\,(r\cos\theta)\,d\theta}{r}$$

$$= \int \frac{r^2\sin^2 d\theta + r^2\cos^2\theta\,d\theta}{r}$$

$$= \int \frac{r^2[\sin^2\theta + \cos^2\theta]\,d\theta}{r} = r\int d\theta = \boxed{2\pi r}$$

This should not be a surpise! Think about what you are doing before you get lost in details of the calculation