USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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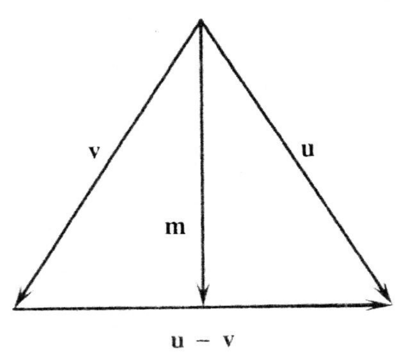
PROBLEM SET I
(due Friday, June 26, 2020)

Problem 1

Show that the two vectors $\vec{u}$ = (3, 4) and $\vec{v}$ = (4, -3) are orthogonal. Draw these two vectors in a cartesian coordinate system.

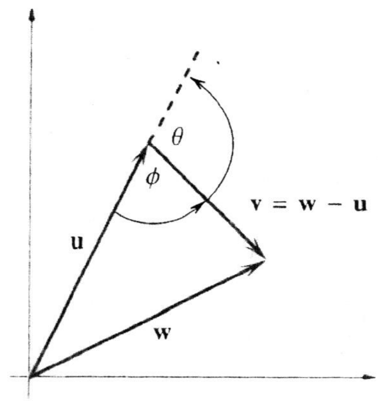
Problem 2

Prove that the line from the apex of an isosceles triangle that bisects its base is perpendicular to the base.



Problem 3

Given the two vectors $\vec{u} = \hat{\imath} + 2\hat{\jmath}$ and $\vec{w} = 2\hat{\imath} + \hat{\jmath}$, determine the angle ϕ in the figure below.



Problem 4

Find the angles between the two vectors $\vec{u} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\vec{v} = 2\hat{\imath}$ - $3\hat{\jmath}$ - $\hat{k}$.

Problem 5

Show that $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$.

Problem 6

Evaluate $\vec{u} \times \vec{v}$ if $\vec{u} = 3\hat{\imath}$ - $\hat{\jmath} + 2\hat{k}$ and $\vec{v} = 2\hat{\imath} + 2\hat{\jmath}$ - $\hat{k}$.

Problem 7

Prove the following property of a vector product

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

Problem 8

Prove the following property of a vector product

$$(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = -c\vec{u} \times \vec{v}$$

Problem 9

Prove the following property of a vector product

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

Problem 10

Prove the the triple scalar product shown below is true

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

Problem 11

Prove that the triple vector product of the following three vectors satisfies the famous rule below

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

Problem 12

Show that the three vectors $\vec{u} = \hat{\imath} + \hat{\jmath}$ - $\hat{k}$, $\vec{v} = 2\hat{\jmath} + \hat{k}$, and $\vec{w} = 2\hat{\imath} + 4\hat{\jmath}$ - $\hat{k}$ are coplanar.

Problem 13

Prove that in plane polar coordinates

$$\hat{\imath} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}$$

$$\hat{\jmath} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}$$

Problem 14

Show that the unit vectors for spherical polar coordinates are given by

$$\hat{\imath} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

$$\hat{\jmath} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$$

$$\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

This coordinate system is called a *spherical coordinate system* because the graph of the equation $r = c =$ constant is a sphere of radius c centered at the origin.

Occassionally we need to know r, θ, and ϕ in terms of x, y, and z. These relations are given by (Problem D–1)

$$r = \left(x^2 + y^2 + z^2\right)^{1/2}$$
$$\cos\theta = \frac{z}{(x^2 + y^2 + z^2)^{1/2}} \tag{D.2}$$
$$\tan\phi = \frac{y}{x}$$

Any point on the surface of a sphere of unit radius can be specified by the values of θ and ϕ. The angle θ represents the declination from the north pole, and hence $0 \le \theta \le \pi$. The angle ϕ represents the angle about the equator, and so $0 \le \phi \le 2\pi$. Although there is a natural zero value for θ (along the north pole), there is none for ϕ. Conventionally, the angle ϕ is measured from the x-axis as illustrated in Figure D.1. Note that r, being the distance from the origin, is intrinsically a positive quantity. In mathematical terms, $0 \le r < \infty$.

In Chapter 6, we will encounter integrals involving spherical coordinates. The differential volume element in Cartesian coordinates is $dxdydz$, but it is not quite so

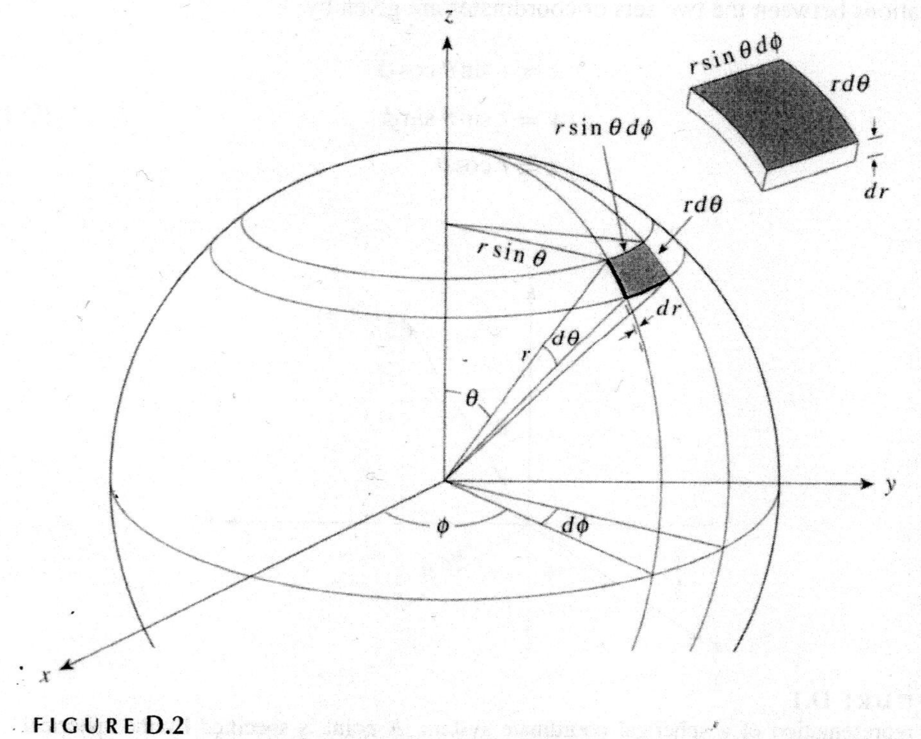


FIGURE D.2
A geometrical construction of the differential volume element in spherical coordinates.

Courtesy of D. A. McQuarrie and J. D. Simon, "Physical chemistry: A molecular approach", (University Science Books), 1997