# USING VECTOR CALCULUS TO SOLVE PROBLEMS IN ELECTRICITY AND MAGNETISM

Summer 2020
Zoom Lecture: F: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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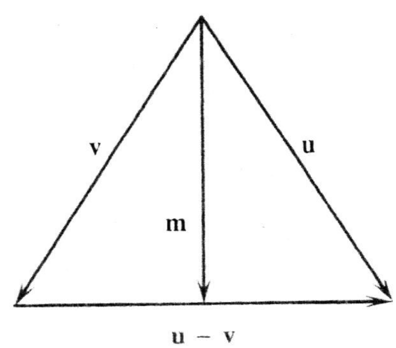
## PROBLEM SET I
(due Friday, June 26, 2020)

### Problem 1

Show that the two vectors $\vec{u} = (3, 4)$ and $\vec{v} = (4, -3)$ are orthogonal. Draw these two vectors in a cartesian coordinate system.

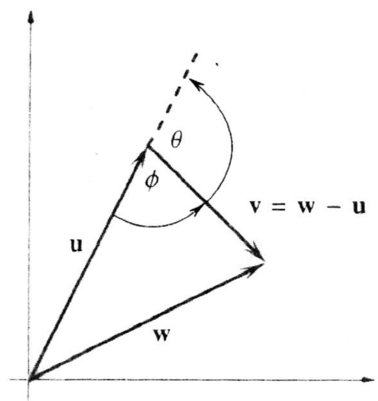
### Problem 2

Prove that the line from the apex of an isosceles triangle that bisects its base is perpendicular to the base.



Problem 3

Given the two vectors $\vec{u} = \hat{\imath} + 2\hat{\jmath}$ and $\vec{w} = 2\hat{\imath} + \hat{\jmath}$, determine the angle $\phi$ in the figure below.




Problem 4

Find the angles between the two vectors $\vec{u} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\vec{v} = 2\hat{\imath}$ - $3\hat{\jmath}$ - $\hat{k}$ .

Problem 5

Show that $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$.

Problem 6

Evaluate $\vec{u} \times \vec{v}$ if $\vec{u} = 3\hat{\imath}$ - $\hat{\jmath} + 2\hat{k}$ and $\vec{v} = 2\hat{\imath} + 2\hat{\jmath}$ - $\hat{k}$ .

Problem 7

Prove the following property of a vector product

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

Problem 8

Prove the following property of a vector product

$$(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = -c\vec{u} \times \vec{v}$$

Problem 9

Prove the following property of a vector product

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

Problem 10

Prove the the triple scalar product shown below is true

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

Problem 11

Prove that the triple vector product of the following three vectors satisfies the famous rule below

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

Problem 12

Show that the three vectors $\vec{u} = \hat{\imath} + \hat{\jmath}$ - $\hat{k}$ , $\vec{v} = 2\hat{\jmath} + \hat{k}$ , and $\vec{w} = 2\hat{\imath} + 4\hat{\jmath}$ - $\hat{k}$ are coplanar.

Problem 13

Prove that in plane polar coordinates

$$\hat{\imath} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}$$

$$\hat{\jmath} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}$$

Problem 14

Show that the unit vectors for spherical polar coordinates are given by

$$\hat{\imath} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

$$\hat{\jmath} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$$

$$\hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

This coordinate system is called a *spherical coordinate system* because the graph of the equation $r = c =$ constant is a sphere of radius $c$ centered at the origin.

Occassionally we need to know $r, \theta$, and $\phi$ in terms of $x, y$, and $z$. These relations are given by (Problem D–1)

$$\begin{aligned} r &= (x^2 + y^2 + z^2)^{1/2} \\ \cos\theta &= \frac{z}{(x^2 + y^2 + z^2)^{1/2}} \\ \tan\phi &= \frac{y}{x} \end{aligned} \tag{D.2}$$

Any point on the surface of a sphere of unit radius can be specified by the values of $\theta$ and $\phi$. The angle $\theta$ represents the declination from the north pole, and hence $0 \le \theta \le \pi$. The angle $\phi$ represents the angle about the equator, and so $0 \le \phi \le 2\pi$. Although there is a natural zero value for $\theta$ (along the north pole), there is none for $\phi$. Conventionally, the angle $\phi$ is measured from the $x$-axis as illustrated in Figure D.1. Note that $r$, being the distance from the origin, is intrinsically a positive quantity. In mathematical terms, $0 \le r < \infty$.

In Chapter 6, we will encounter integrals involving spherical coordinates. The differential volume element in Cartesian coordinates is $dxdydz$, but it is not quite so

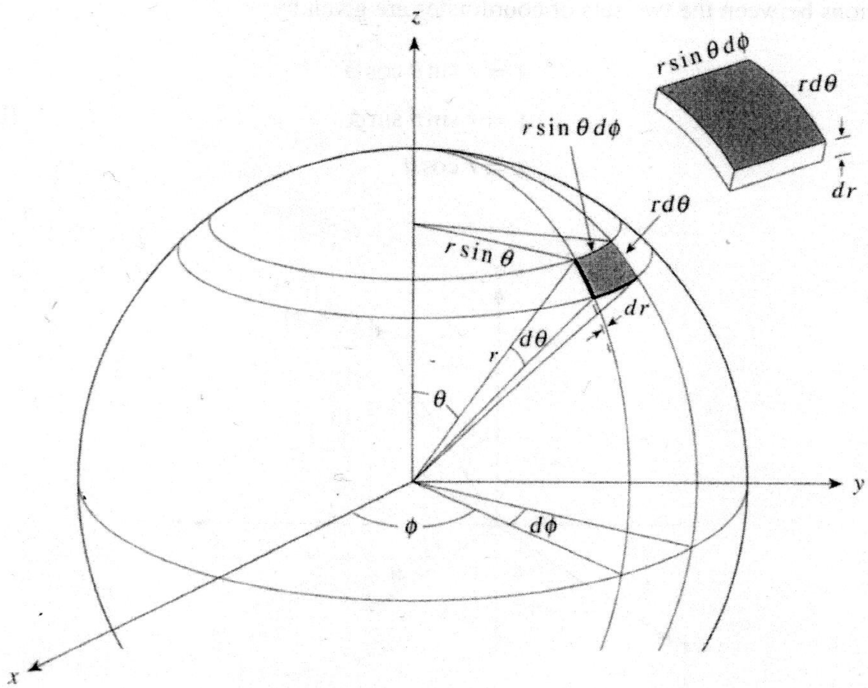



**FIGURE D.2**
A geometrical construction of the differential volume element in spherical coordinates.

Courtesy of D. A. McQuarrie and J. D. Simon, "Physical chemistry: A molecular approach", (University Science Books), 1997

## Problem 1

Show that the two vectors

$$\vec{U} = (3, 4)$$
$$\vec{V} = (4, -3)$$

are orthogonal. Draw these two vectors in a cartesian coordinate system.

Find the dot product of $\vec{U}$ and $\vec{V}$

$$\vec{U} = 3\hat{i} + 4\hat{j}$$
$$\vec{V} = 4\hat{i} - 3\hat{j}$$

$\vec{U} \cdot \vec{V} = 12 - 12 = 0$ Therefore $\vec{U} \perp \vec{V}$ since

$$\vec{U} \cdot \vec{V} = \|\vec{U}\| \|\vec{V}\| \cos\theta = 0$$

if $\theta = \frac{\pi}{2}$ provided

$\|\vec{U}\| \neq 0$

$\|\vec{V}\| \neq 0$

## Problem 2

Prove that the line from the apex of an isosceles triangle that bisects its base is perpendicular to the base.

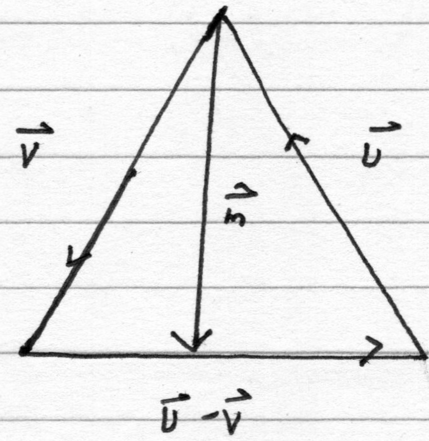



Fig. 2.1

We are given Fig. 2.1 as a pictorial aid to solve this problem. Clearly

$$\vec{m} + \frac{1}{2}(\vec{U} - \vec{V}) = \vec{U}$$

$$\vec{m} - \frac{1}{2}(\vec{U} - \vec{V}) = \vec{V}$$

or combining the two

$$2\vec{m} + \frac{1}{2}\vec{V} - \frac{1}{2}\vec{V} = \vec{U} + \vec{V}$$

$$2\vec{m} + 0\vec{V} = \vec{U} + \vec{V}$$

$$\boxed{\vec{m} = \frac{1}{2}(\vec{U} + \vec{V})}$$

The base is represented by $\vec{U} - \vec{V}$
and

$$\vec{m} \cdot (\vec{U} - \vec{V}) = \frac{1}{2}(\vec{U} + \vec{V}) \cdot (\vec{U} - \vec{V})$$

$$= \frac{1}{2}\left(\|\vec{U}\|^2 - \|\vec{V}\|^2\right)$$

Since our triangle is an isosceles triangle

$$\|\vec{U}\|^2 = \|\vec{V}\|^2$$

so

$\vec{m} \cdot (\vec{U} - \vec{V}) = 0$ and $\vec{m} \perp$ to the base of the triangle.

Q.E.D.

<u>Problem 3</u>

<u>Given the two vectors $\vec{U} = \hat{i} + 2\hat{j}$</u>
<u>and $\vec{W} = 2\hat{i} + \hat{j}$, determine the angle in the</u>
<u>figure below</u>

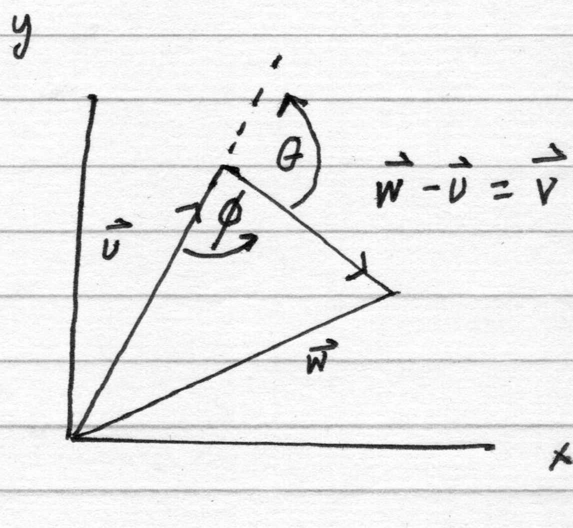



Fig. 3-1

The angle $\theta$ between the vectors $\vec{U}$ and $\vec{V}$ is defined when they are arranged tail-to-tail

$$(3\text{-}1) \qquad \vec{U}\cdot\vec{V} = \|\vec{U}\|\,\|\vec{V}\|\cos\theta$$

or

$$(3\text{-}2) \qquad \cos\theta = \frac{\vec{U}\cdot\vec{V}}{\|\vec{U}\|\,\|\vec{V}\|}$$

From Fig. 3.1 this angle is

$$(3\text{-}3) \qquad \theta = \pi - \phi \,.$$

Using $\vec{V} = \vec{W} - \vec{U} = \hat{\imath} - \hat{\jmath}$ in Eq. (3-1), we have

$$(3\text{-}4) \qquad \vec{U}\cdot\vec{V} = \vec{U}\cdot(\hat{\imath} - \hat{\jmath}) = (\hat{\imath} + 2\hat{\jmath})\cdot(\hat{\imath} - \hat{\jmath})$$

$$(3\text{-}5) \qquad \vec{U}\cdot\vec{V} = 1 - 2 = -1 = \|\vec{U}\|\,\|\vec{V}\|\cos\theta$$

$$\cos\theta = \frac{-1}{\|\vec{U}\|\,\|\vec{V}\|} = \frac{-1}{\sqrt{5}\,\sqrt{5}} = \frac{1}{\sqrt{10}}$$

or

$$\theta = 108.4^\circ$$

Thus $\phi = \pi - \theta$ and $\boxed{\phi = 71.6^\circ}$

## Problem 4

Find the angles between the two vectors

$$\vec{U} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{V} = 2\hat{i} - 3\hat{j} - \hat{k}$$

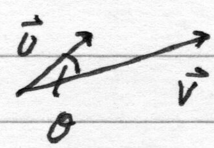
Note

$$\vec{U}\cdot\vec{V} = 2 - 6 - 3 = -7$$

$$\|\vec{U}\| = (1+4+9)^{1/2} = \sqrt{14}$$

$$\|\vec{V}\| = (4+9+1)^{1/2} = \sqrt{14}$$

$$\vec{U}\cdot\vec{V} = \|\vec{U}\|\,\|\vec{V}\|\cos\theta$$




Not drawn to scale

$$\cos\theta = \frac{\vec{U}\cdot\vec{V}}{\|\vec{U}\|\,\|\vec{V}\|} = \frac{-7}{14} = -\frac{1}{2}$$

or

$$\boxed{\theta = \frac{2\pi}{3} = 120^\circ}$$

## Problem 5

Show that $\vec{U} \times \vec{V}$ is perpendicular to $\vec{U}$.

$$\vec{U} \times \vec{V} = \hat{i}\,(U_y V_z - U_z V_y) + \hat{j}\,(U_z V_x - U_x V_z) + \hat{k}\,(U_x V_y - U_y V_x)$$

Let us evaluate

$$\vec{U} \cdot (\vec{U} \times \vec{V}) = U_x\,(U_y V_z - U_z V_y) + U_y\,(U_z V_x - U_x V_z) + U_z\,(U_x V_y - U_y V_x)$$

Expanding this out gives

$$\vec{U} \cdot (\vec{U} \times \vec{V}) = \overset{①}{U_x U_y V_z} - \overset{②}{U_x U_z V_y} + \overset{③}{U_y U_z V_z} - \overset{④}{U_y U_x V_z} + \overset{⑤}{U_z U_x V_y} - \overset{⑥}{U_z U_y V_x}$$

Where we ~~see~~ see terms ① and ④ cancel out, terms ② and ⑤ cancel out, and terms ③ and ⑥ cancel out.

Thus $\vec{U} \cdot (\vec{U} \times \vec{V}) = 0$

Note that the product $\vec{V} \cdot (\vec{U} \times \vec{V}) = 0$ also.
(Can you see this with doing the algebra)?

## Problem 6

**Evaluate $\vec{U} \times \vec{V}$ if**

$$\vec{U} = 3\hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{V} = 2\hat{i} + 2\hat{j} - \hat{k}.$$

$$\vec{U} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ U_x & U_y & U_z \\ V_x & V_y & V_z \end{vmatrix} = \hat{i}\,(U_y V_z - U_z V_y) + \hat{j}\,(U_z V_x - U_x V_z)$$

$$+ \hat{k}\,(U_x V_y - U_y V_x)$$

$$\vec{U} \times \vec{V} = \hat{i}\,\big((-1)(-1) - (2)(2)\big)$$

$$+ \hat{j}\,(2 \cdot 2 - 3(-1))$$

$$+ \hat{k}\,(3 \cdot 2 - (-1)(+2))$$

$$= \hat{i}\,(1 - 4) + \hat{j}\,(4 + 3) + \hat{k}\,(6 + 2)$$

$$= -3\hat{i} + 7\hat{j} + 8\hat{k}$$

Note that $\vec{U} \times \vec{V}$ is perpendicular to both $\vec{U}$ and $\vec{V}$.

## Problem 7

Prove the following property of a vector product

$$\vec{U} \times \vec{V} = -\vec{V} \times \vec{U}$$

(7-1) $\vec{U} \times \vec{V} = \hat{i}\,(U_y V_z - U_z V_y) + \hat{j}\,(U_z V_x - U_x V_z)$
$+ \hat{k}\,(U_x V_y - U_y V_x)$

Pulling out a minus sign gives

(7-2) $\vec{U} \times \vec{V} = -\hat{i}\,(U_z V_y - U_y V_z) - \hat{j}\,(U_x V_z - U_z V_x)$
$-\hat{k}\,(U_y V_x - U_x V_y)$

which can be rewritten as

(7-3) $\vec{U} \times \vec{V} = -\hat{i}\,(V_y U_z - V_z U_y) - \hat{j}\,(V_z U_x - V_x U_z)$
$-\hat{k}\,(V_x U_y - V_y U_x)$

since scaler multiplication is commutative

Equation (7-3) becomes

$$\boxed{(7\text{-}4)\quad \vec{U} \times \vec{V} = -(\vec{V} \times \vec{U})}$$

Q.E.D.

## Problem 8

Prove the following property of a vector product

$$(c\vec{U}) \times \vec{V} = -\vec{U} \times (c\vec{V}) = -c\vec{U} \times \vec{V}$$

$$(c\vec{U}) \times \vec{V} = \hat{i}\,(cU_y V_z - cU_z V_y)$$

$$+\hat{j}\,(cU_z V_x - cU_x V_z)$$

$$+\hat{k}\,(cU_x V_y - cU_y V_x)$$

$$= \hat{i}\,(U_y (cV_z) - U_z (cV_y))$$

$$+\hat{j}\,(U_z (cV_x) - U_x (cV_z))$$

$$+\hat{k}\,(U_x (cV_y) - U_y (cV_x))$$

$$= \vec{U} \times (c\vec{V})$$

$$= c\,[\hat{i}(U_y V_z - U_z V_y) + \hat{j}\,(U_z V_x - U_x V_z)$$

$$+\hat{k}\,(U_x V_y - U_y V_x)]$$

$$= c\,(\vec{U} \times \vec{V}) = c\,\vec{U} \times \vec{V}$$

Q.E.D.

Problem 9

Prove the following property of a vector-product

$$\vec{U} \times (\vec{V} + \vec{W}) = \vec{U} \times \vec{V} + \vec{U} \times \vec{W}$$

$$\vec{V} + \vec{W} = \hat{i}\,(V_x + W_x) + \hat{j}\,(V_y + W_y) + \hat{k}\,(V_z + W_z)$$

$$\vec{U} \times (\vec{V} + \vec{W}) = \hat{i}\left[U_y\,(\vec{V} + \vec{W})_z - U_z\,(\vec{V} + \vec{W})_y\right]$$

$$+ \hat{j}\left[U_z\,(\vec{V} + \vec{W})_x - U_x\,(\vec{V} + \vec{W})_z\right]$$

$$+ \hat{k}\left[U_x\,(\vec{V} + \vec{W})_y - U_y\,(\vec{V} + \vec{W})_x\right]$$

$$= \hat{i}\left[U_y\,(V_z + W_z) - U_z\,(V_y + W_y)\right]$$

$$+ \hat{j}\left[U_z\,(V_x + W_x) - U_x\,(V_z + W_z)\right]$$

$$= \hat{k}\left[U_x\,(V_y + W_y) - U_y\,(V_x + W_x)\right]$$

Upon regrouping

$$\vec{U} \times (\vec{V} + \vec{W}) = \hat{i}\left[U_y V_z - U_z V_y + U_y W_z - U_z W_y\right]$$

$$+ \hat{j}\left[U_z V_x - U_x V_z + U_z W_x - U_x W_z\right]$$

$$+ \hat{k}\left[U_x V_y - U_y V_x + U_x W_y - U_y W_x\right]$$

Sorting out terms yields where we see what we are trying to show

$$\vec{U} \times (\vec{V} + \vec{W}) = \hat{i}\,(U_y V_z - U_z V_y] + \hat{j}\,[U_z V_x - U_x V_z] + \hat{k}\,[U_x V_y - U_y V_x]$$

$$+ \hat{i}\,[U_y W_z - U_z W_y] + \hat{j}\,[U_z W_x - U_x W_z] + \hat{k}\,[U_x W_y - U_y W_x]$$

which is simply

$$\boxed{\vec{U} \times (\vec{V} + \vec{W}) = \vec{U} \times \vec{V} + \vec{U} \times \vec{W}}$$

Q.E.D.

<u>Problem 10</u>

<u>Prove the triple scalar product shown below is true.</u>

$$\vec{V} \times \vec{W} = \hat{i}\,(V_y W_z - V_z W_y) + \hat{j}\,(V_z W_x - V_x W_z) + \hat{k}\,(V_x W_y - V_y W_x)$$

$$\vec{U} \cdot (\vec{V} \times \vec{W}) = \underset{1}{U_x V_y W_z} - \underset{2}{U_x V_z W_y} + \overset{3}{U_y V_z W_x} - \overset{4}{U_y V_x W_z} + \underset{5}{U_z V_x W_y} - \underset{6}{U_z V_y W_x}$$

Let us factor out $W_x$ from terms ③ and ⑥, $W_y$ from terms ② and ⑤; and $W_z$ from terms ① and ④ to yield

$$\vec{U}\cdot(\vec{V}\times\vec{W}) = W_x\left[U_y V_z - U_z V_y\right]$$

$$+ W_y\left[U_z V_x - U_x V_z\right]$$

$$+ W_z\left[U_x V_y - U_y V_x\right]$$

$$= \vec{W}\cdot(\vec{U}\times\vec{V}) = (\vec{U}\times\vec{V})\cdot\vec{W}$$

since the scalar product is commutative.

Q.E.D.

## Problem 12

Show that the triple vector product of the following three vectors satisfies the famous rule below

$$\vec{U} \times (\vec{V} \times \vec{W}) = (\vec{U} \cdot \vec{W})\vec{V} - (\vec{U} \cdot \vec{V})\vec{W}$$

Let us start with

(12-1) $$\vec{V} \times \vec{W} = \hat{i}\,(V_y W_z - V_z W_y) + \hat{j}\,(V_z W_x - V_x W_z) + \hat{k}\,(V_x W_y - V_y W_x)$$

Now

(12-2) $$\vec{U} \times (\vec{V} \times \vec{W}) = \hat{i}\left(U_y (\vec{V} \times \vec{W})_z - U_z (\vec{V} \times \vec{W})_y\right) + \hat{j}\left(U_z (\vec{V} \times \vec{W})_x - U_x (\vec{V} \times \vec{W})_z\right) + \hat{k}\left(U_x (\vec{V} \times \vec{W})_y - U_y (\vec{V} \times \vec{W})_x\right)$$

Expanding (12-2) out gives

(12-3)

$$\vec{U} \times (\vec{V} \times \vec{W}) = \hat{\imath}\left[ U_y \left( \overset{(1)}{V_x W_y} - \overset{(2)}{V_y W_x} \right) - U_z \left( \overset{(3)}{V_z W_x} - \overset{(4)}{V_x W_z} \right) \right]$$

$$+ \hat{\jmath}\left[ U_z \left( \overset{(5)}{V_y W_z} - \overset{(6)}{V_z W_y} \right) - U_x \left( \overset{(7)}{V_x W_y} - \overset{(8)}{V_y W_x} \right) \right]$$

$$+ \hat{k}\left[ U_x \left( \overset{(9)}{V_z W_x} - \overset{(10)}{V_x W_z} \right) - U_y \left( \overset{(11)}{V_y W_z} - \overset{(12)}{V_z W_y} \right) \right]$$

Let us think about our end goal. It contains a vector $\vec{V}$ which is isolated all by itself so let us factor $V_x$ from (1) and (4), $V_y$ from (5) and (8), and $V_z$ from (9) and (12) to yield

(12-4)

$$\vec{U} \times (\vec{V} \times \vec{W}) = \hat{\imath}\, V_x \left[ \underset{(1)}{U_y W_y} + \underset{(4)}{U_z W_z} \right] + \hat{\jmath}\, V_y \left[ \underset{(5)}{U_z W_z} + \underset{(8)}{U_x W_x} \right]$$

$$+ \hat{k}\, V_z \left[ \underset{(9)}{U_x W_x} + \underset{(12)}{U_y W_y} \right] - \underset{(2)}{\hat{\imath}\, U_y V_y W_x} - \underset{(3)}{\hat{\imath}\, U_z V_z W_x}$$

$$- \underset{(6)}{\hat{\jmath}\, U_z V_z W_y} - \underset{(7)}{\hat{\jmath}\, U_x V_x W_y} - \underset{(10)}{\hat{k}\, U_x V_x W_z}$$

$$- \underset{(11)}{\hat{k}\, U_y V_y W_z}$$

Now look at the first six terms (1, 4, 5, 8, 9 and 12) we could convert them all to $\vec{V}(\vec{U}\cdot\vec{W})$ if we add and subtract $\hat{i}V_xU_xW_x$, $\hat{j}V_yU_yW_y$, and $\hat{k}V_zU_zW_z$ to (12-4). This now becomes

(12-5) $$\vec{U}\times(\vec{V}\times\vec{W}) = \vec{V}(\vec{U}\cdot\vec{W}) - \hat{i}V_xU_xW_x \quad ①$$

$$-\hat{j}V_yU_yW_y \quad ② \quad -\hat{k}V_zU_zW_z \quad ③ \quad -\hat{i}U_yV_yW_x \quad ④ \quad -\hat{i}U_zV_zW_x \quad ⑤$$

$$-\hat{j}U_zV_zW_y \quad ⑥ \quad -\hat{j}V_xU_xW_y \quad ⑦ \quad -\hat{k}U_xV_xW_z \quad ⑧ \quad -\hat{k}U_yV_yW_z \quad ⑨$$

We use the same trick again!

Let us factor $W_x$ from ①, ④ and ⑤, $W_y$ from ②, ⑥ and ⑦, and $W_z$ from ③, ⑧ and ⑨ to yield from (12-5)

(12-6) $$\vec{U}\times(\vec{V}\times\vec{W}) = -[W_x(\vec{U}\cdot\vec{V})\hat{i} + W_y(\vec{U}\cdot\vec{V})\hat{j} + W_z(\vec{U}\cdot\vec{V})\hat{k}] + \vec{V}(\vec{U}\cdot\vec{W})$$

or

(12-7) $$\vec{U}\times(\vec{V}\times\vec{W}) = -[(\vec{U}\cdot\vec{V})\vec{W}] + \vec{V}(\vec{U}\cdot\vec{W})$$

or

(12-8) $$\boxed{\vec{U}\times(\vec{V}\times\vec{W}) = \vec{V}(\vec{U}\cdot\vec{W}) - \vec{W}(\vec{U}\cdot\vec{V})}$$

Q.E.D.

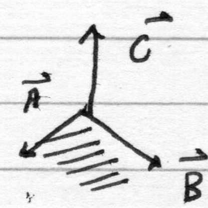
## Problem 12

Show that the three vectors

$$\boxed{\begin{aligned}\vec{U} &= \hat{i} + \hat{j} - \hat{k}\\ \vec{V} &= 2\hat{j} + \hat{k}\\ \vec{W} &= 2\hat{i} + 4\hat{j} - \hat{k}\end{aligned}}$$

are coplanar.

If $\vec{A} \times \vec{B} = \vec{C}$




and if

$$\vec{C} \cdot (\vec{A} \times \vec{B}) = 0$$

then $\vec{C}$ can not be $\perp$ plane containing $\vec{A}$ and $\vec{B}$ and $\vec{A}, \vec{B}$, and $\vec{C}$ must all be coplanar

$$\vec{V} \times \vec{W} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 1 \\ 2 & 4 & -1 \end{vmatrix} = -2\hat{i} + 2\hat{j} + 0\hat{k} - 4\hat{k} - 0\hat{j} - 4\hat{i}$$

$$= -6\hat{i} + 2\hat{j} - 4\hat{k}$$

$$\vec{U} \cdot (\vec{V} \times \vec{W}) = (\hat{i} + \hat{j} - \hat{k}) \cdot (-6\hat{i} + 2\hat{j} - 4\hat{k})$$

$$= -6 + 2 + 4 = 0$$

Thus $\vec{U}, \vec{V}$, and $\vec{W}$ are coplanar

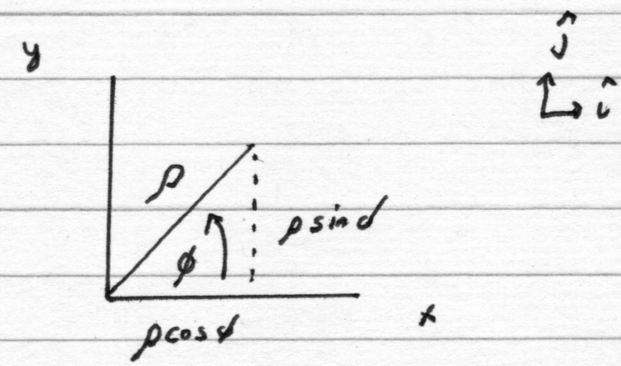
## Problem 13

Prove that in plane polar coordinates

$$\hat{\imath} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}$$

$$\hat{\jmath} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}.$$

Let us start with plane polar coordinates

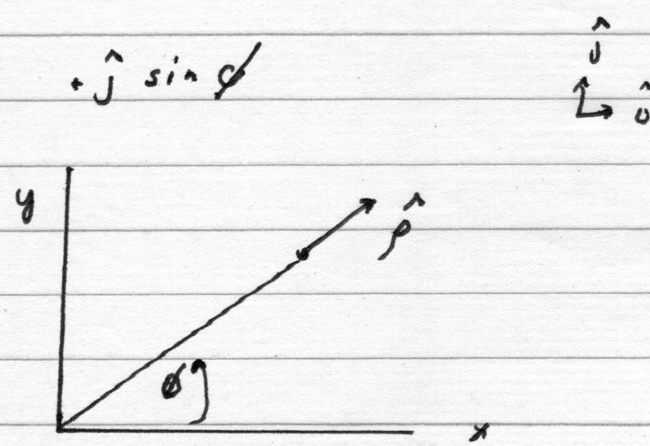



$$\vec{\rho} = \hat{\imath}\,\rho\cos\phi + \hat{\jmath}\,\rho\sin\phi$$

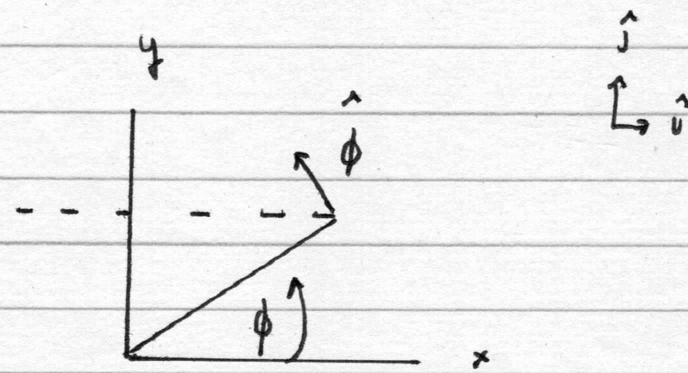
$$\vec{\rho} = \hat{\rho}\,\rho$$

So $\hat{\rho} = \hat{\imath}\cos\phi + \hat{\jmath}\sin\phi$

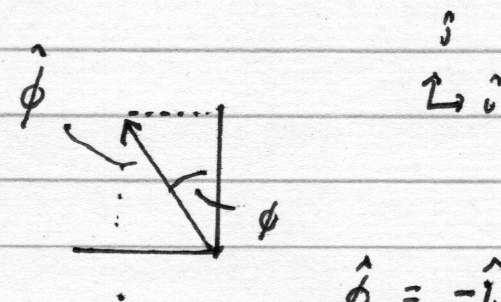
and $\hat{\rho}$ is illustrated here. Check that $\hat{\rho}$ makes sense for $\phi = 0, \pi/2$



What is $\hat{\phi}$ ?




Blow up picture and resolve $\phi$




$$\hat{\phi} = -\hat{\imath}\sin\phi + \hat{j}\cos\phi$$

Check limiting behavior for $\phi = 0, \frac{\pi}{2}$

Thus

$$\boxed{\begin{aligned} &(13\text{-}1) \quad \hat{\rho} = \hat{\imath}\cos\phi + \hat{j}\sin\phi \\ &(13\text{-}2) \quad \hat{\phi} = -\hat{\imath}\sin\phi + \hat{j}\cos\phi \end{aligned}}$$

in plane polar coordinates

Now use (13-1) and (13-2) to find $\hat{i}, \hat{j}$

Multiply (13-1) by $\cos\phi$

(13-3) $\cos\phi\,\hat{\rho} = \hat{i}\cos^2\phi + \hat{j}\sin\phi\cos\phi$

Multiply (13-2) by $-\sin\phi$

(13-4) $-\sin\phi\,\hat{\phi} = \hat{i}\sin^2\phi - \hat{j}\sin\phi\cos\phi$

and add (13-3) + (13-4) to get

$$\boxed{(13\text{-}5)\quad \hat{i} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}}$$

Similarly multiply (13-1) by $\sin\phi$

(13-5) $\hat{\rho}\sin\phi = \hat{i}\sin\phi\cos\phi + \hat{j}\sin^2\phi$

and multiply (13-2) by $\cos\phi$

(13-6) $\hat{\phi}\cos\phi = -\hat{i}\sin\phi\cos\phi + \hat{j}\cos^2\phi$

and combine (13-5) and (13-6) to yield

$$\boxed{(13\text{-}7)\quad \hat{j} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}}$$

Problem 14

Prove that in spherical polar coordinates

(14-1) $\hat{i} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$

(14-2) $\hat{j} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$

(14-3) $\hat{k} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

Let us start with the unit vectors for spherical polar coordinates

(14-4) $\hat{r} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$

(14-5) $\hat{\theta} = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta$

(14-6) $\hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi$

The idea is to use these equations to find $\hat{i}, \hat{j}$, and $\hat{k}$

Let us first multiply (14-4) by $\sin\theta$ to get

(14-7) $\hat{r}\sin\theta = +\hat{\imath}\sin^2\theta\cos\phi + \hat{\jmath}\sin^2\theta\sin\phi + \hat{k}\sin\theta\cos\theta$

and then multiply (14-5) by $\cos\theta$ to get

(14-8) $\hat{\theta}\cos\theta = \hat{\imath}\cos^2\theta\cos\phi + \hat{\jmath}\cos^2\theta\sin\phi - \hat{k}\sin\theta\cos\theta$

Upon adding (14-7) and (14-8) we obtain

(14-9) $\hat{r}\sin\theta + \hat{\theta}\cos\theta = \hat{\imath}\cos\phi + \hat{\jmath}\sin\phi$

Now let us multiply (14-9) by $\sin\phi$ to yield

(14-10) $\hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi =$

$\hat{\imath}\sin\phi\cos\phi + \hat{\jmath}\sin^2\phi$

and then multiply (14-6) by $\cos\phi$ to obtain

(14-11) $\hat{\phi}\cos\phi = -\hat{\imath}\sin\phi\cos\phi + \hat{\jmath}\cos^2\phi$

Upon adding (14-10) and (14-11) we get

(14-12) $\hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$

$= \hat{\jmath}$

or

$$\boxed{(14\text{-}13) \qquad \hat{j} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi}$$

which is one down and two to go!

Next let us multiply (14-9) by $\cos\phi$ to obtain

$$(14\text{-}14) \qquad \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi = \hat{i}\cos^2\phi + \hat{j}\sin\phi\cos\phi$$

and then multiply (14-6) by $\sin\phi$ to yield

$$(14\text{-}15) \qquad \hat{\phi}\sin\phi = -\hat{i}\sin^2\phi + \hat{j}\sin\phi\cos\phi .$$

Upon subtracting (14-14) − (14-15) we get

$$(14\text{-}16) \qquad \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi = \hat{i}$$

or

$$(14\text{-}17) \qquad \boxed{\hat{i} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi}$$

Which is two down and one to go!

Finally, let us multiply (14-4) by $\cos\theta$ to obtain

(14-18) $$\hat{r}\cos\theta = \hat{i}\sin\theta\cos\theta\cos\phi + \hat{j}\sin\theta\cos\theta\sin\phi + \hat{k}\cos^2\theta$$

and multiply (14-5) by $-\sin\theta$ to yield

(14-19) $$-\sin\theta\,\hat{\theta} = -\hat{i}\sin\theta\cos\theta\cos\phi - \hat{j}\sin\theta\cos\theta\sin\phi + \hat{k}\sin^2\theta$$

If we add (14-18) and (14-19) we get

(14-20) $$\hat{r}\cos\theta - \hat{\theta}\sin\theta = \hat{k}\left[\sin^2\theta + \cos^2\theta\right]$$

or

$$\boxed{\hat{k} = \hat{r}\cos\theta - \hat{\theta}\sin\theta}$$

Q.E.D.