AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021
Zoom Lecture: Tu: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET XI
(due Tuesday, August 3, 2021)

Problem 1

(a) We continue here with our discussion of the coupled harmonic oscillator from Lecture 11 where we showed that the equations of motion for the two masses were given by

$$F_n = m\,\ddot{u}_n = -2\alpha u_n + \alpha u_{n+1}$$

$$F_{n+1} = m\,\ddot{u}_{n+1} = -2\alpha u_{n+1} + \alpha u_n$$

If we assume solutions of the form for these two equations

$$u_n = A_n\,e^{i\omega t}$$

$$u_{n+1} = A_{n+1}\,e^{i\omega t}$$

where we will assume that ω is the frequency of our system. Show that you can obtain the following real equations

$$-m\omega^2 A_n = -2\alpha\,A_n + \alpha\,A_{n+1}$$

$$-m\omega^2 A_{n+1} = -2\alpha\,A_{n+1} + \alpha\,A_n.$$

(b) Show that we can recast these equations in the form of an eigenvalue problem

$$A\vec{v} = \lambda\vec{v}$$

where

$$\mathrm{A} = \begin{pmatrix} 2\alpha & -\alpha \\ -\alpha & 2\alpha \end{pmatrix}$$

$$\vec{\mathbf{v}} = \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix}$$

and

$$\lambda = m\omega^2.$$

(c) Solve the corresponding characteristic equation to discover that our coupled harmonic oscillator problem now has **two** distinct eigenvalues or frequencies

$$\omega_1 = \sqrt{\frac{\alpha}{m}}$$

$$\omega_2 = \sqrt{\frac{3\alpha}{m}}$$

(d) For each eigenvalue or frequency solve for the corresponding eigenvector to yield

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} a \\ a \end{pmatrix}$$

and

$$\vec{\mathbf{v}}_2 = \begin{pmatrix} a \\ -a \end{pmatrix}$$

where **a** is an arbitrary constant.

(e) Now return to the original assumptions for our solutions in Part 1(a). Having solved our eigenvalue problem for the mass at position **n**

$$u_n = A_n\, e^{i\omega t}$$

we discovered that there are **two** allowed frequencies thus giving two different solutions

$$u_n = A_n\, e^{i\omega_1 t} = a\, e^{i\omega_1 t}$$

and

$$u_n = A_n\, e^{i\omega_2 t} = a\, e^{i\omega_2 t}$$

We can express the general solution to our original differential equation as a linear combination of these two solutions

$$u_n = c_1 a\, e^{i\omega_1 t} + c_2 a\, e^{i\omega_2 t}$$

where c_1 and c_2 are arbitrary constants.

The same line of reasoning applies to the mass at position **n+1**

$$u_{n+1} = A_{n+1}\, e^{i\omega t}$$

that there are **two** allowed frequencies again thus yielding two different solutions

$$u_{n+1} = A_{n+1}\, e^{i\omega_1 t} = a\, e^{i\omega_1 t}$$

and

$$u_{n+1} = A_{n+1}\, e^{i\omega_2 t} = -a\, e^{i\omega_2 t}$$

We can also express the general solution to our original differential equation as a linear combination of these two solutions

$$u_{n+1} = c_1 a\, e^{i\omega_1 t} - c_2 a\, e^{i\omega_2 t}$$

where c_1 and c_2 are arbitrary constants.

If we define new constants

$$b_1 = ac_1$$

$$b_2 = ac_2$$

we obtain two simplified equations

$$\boldsymbol{u_n = b_1\, e^{i\omega_1 t} + b_2\, e^{i\omega_2 t}}$$

and

$$\boldsymbol{u_{n+1} = b_1\, e^{i\omega_1 t} - b_2\, e^{i\omega_2 t}}$$

(f) Now what are these equations in Part 1(e) telling us? We will now show that for $\boldsymbol{b_2 = 0}$ and $\boldsymbol{b_1 \neq 0}$ our two masses will vibrate in phase. Let us introduce an arbitrary phase $\boldsymbol{\phi_1}$ to the problem such that

$$\boldsymbol{b_1 = A_1\, e^{-i\phi_1}}$$

Show that our two equations now become

$$\boldsymbol{u_n(t) = A_1 \cos\left(\omega_1 t - \phi_1\right)}$$

$$\boldsymbol{u_{n+1}(t) = A_1 \cos\left(\omega_1 t - \phi_1\right)}$$

which shows that the two masses vibrate in unison.

(g) We will next consider what our two equations in Problem 1(e) are telling us for the case where $\boldsymbol{b_1 = 0}$ and $\boldsymbol{b_2 \neq 0}$. Here our two masses will vibrate in opposite directions. Let us introduce an arbitrary phase $\boldsymbol{\phi_2}$ to the problem such that

$$\boldsymbol{b_2 = A_2\, e^{-i\phi_2}}$$

Show that our two equations from Problem 1(e) now become

$$\boldsymbol{u_n(t) = A_2 \cos\left(\omega_2 t - \phi_2\right)}$$

$$\boldsymbol{u_{n+1}(t) = -A_2 \cos\left(\omega_2 t - \phi_2\right) = A_2 \cos\left(\omega_2 t - \phi_2 + \pi\right)}$$

which means that our masses vibrate out of phase by 180^o and hence in opposite directions. Our final results from Problem 1(f) and Problem 1(g) tell us the **normal modes** of vibration for our coupled harmonic oscillator system. Of course the real system will be described by a superposition of these two normal modes of vibration. (*Cf.* Dr. H. Trevor Johnson-Steigelman, The University of Maryland at College Park, (http://youtu.be/JOSK7Zy8iko)

Problem 2

Diagonalize the following matrix $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{3} \end{pmatrix}$$

Problem 3

Diagonalize the following matrix $\mathbf{Z}$:

$$\mathbf{Z} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Problem 4

Diagonalize the following matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{pmatrix} \mathbf{2} & \mathbf{5} \\ \mathbf{1} & \mathbf{6} \end{pmatrix}$$

Problem 5

Diagonalize the following matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{pmatrix}$$

Problem 6

Diagonalize the following matrix $\mathbf{F}$:

$$\mathbf{F} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Problem 7

Diagonalize the following matrix $\mathbf{K}$:

$$\mathbf{K} = \begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} \end{pmatrix}$$

Problem 8

Diagonalize the following matrix $\mathbf{J}$:

$$\mathbf{J} = \begin{pmatrix} \mathbf{3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \end{pmatrix}$$

Problem 9

Diagonalize the following matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} & \mathbf{2} \end{pmatrix}$$

Problem 10

Please look at the absolutely beautiful animated lecture by Grant Sanderson "Linear Transformations and Matrices, Essence of Linear Algebra: Chapter 3 (10:58 minutes)." His geometric discussion of the basic concepts in linear algebra is so spectacular and I could never accomplish what he does in a lecture. Learning in the 21^{th} century is accomplished through all sorts of mechanisms (e.g. lectures, problem sets, reading, recitation sections, videos, etc.) so we should take advantage of all of these approaches!

In particular, in this video he shows how matrices are examples of things called **"linear transformations"** in a very clear and geometric way! I highly suggest looking at it once you have mastered Lecture 9 and Problem Set 9.

Python Exercise 11

1. The following is a Python script (a fancy word for a program) for diagonalizing a matrix. Note the similarity with the Python script for finding the eigenvalues and eigenvectors of a matrix.

import numpy as np

A = np.array([[1,2],[2,1]])#defining the matrix A

print(A)#print the original matrix A

L,V =np.linalg.eig(A)# Note that the function **np.linalg.eig()** returns the eigenvalues in a vector and the eigenvectors in a matrix. Here L stands for the eigenvalues (Lambda) and V stands for the eigenvectors (Vectors)

print(L)#print the eigenvalues of A

print(V)#print the eigenvectors of A

D = np.diag(L)#Note that the function **np.diag()** takes the eigenvalues L and forms a diagonal matrix **D**.

print(D)#print this diagonal matrix D generated from above

2. Create the following random matrices of sizes: **(5 x 5)**, **(10 x 10)**, **(20 x 20)**, and **(50 x 50)**. Use Python to find their eigenvalues and eigenvectors. In the above script the diagonal matrix is simply built from the eigenvalues. Modify this script to explicitly compute the diagonal matrix **D** using the expression

$$\boldsymbol{D = S^{-1}AS}$$

3. In the beginning of this course we introduced the concept of the **rank** of a matrix to provide a neat way of determining if a set of linear equations had a unique solution, an infinite number of solutions, or no solution. This definition of the **rank** of a matrix involved calculating determinants which was something we gained expertise in. Actually it turns out that this is just one definition for the **rank** of a matrix and the concept of the **rank** of a matrix is very powerful and it has many definitions.

Perhaps the most important definition of the **rank** of a matrix is the maximum number of column vectors it has which are linearly independent. Obviously we could not use this definition earlier in the course as we did not know what linear independence meant! Well that was then and this is now.

Calculating the **rank** of a large matrix is quite an involved thing. Here we see how Python does it using the **np.linalg.matrix _rank()** function

import numpy as np

P = np.array([[1,2,],[2,1]])

print(P)

r1 = np.linalg.matrix_rank(P)

print(r1)

Go back to Problem Set V (Problems 1, 2, and 3) and verify your results for both the coefficient and augmented matrices.

Problem 1

(a) $$m\ddot{U}_n = -2\alpha U_n + \alpha U_{n+1}$$

$$U_n = A_n e^{i\omega t}$$

$$\dot{U}_n = i\omega A_n e^{i\omega t} \quad ; \quad \ddot{U}_n = -\omega^2 A_n e^{i\omega t}$$

$$U_{n+1} = A_{n+1} e^{i\omega t}$$

$$m\left(-\omega^2 A_n e^{i\omega t}\right) = -2\alpha A_n e^{i\omega t} + \alpha A_{n+1} e^{i\omega t}$$

Upon cancelling out terms involving $e^{i\omega t}$ we get

$$\boxed{-m\omega^2 A_n = -2\alpha A_n + \alpha A_{n+1}}$$

$$m\ddot{U}_{n+1} = -2\alpha U_{n+1} + \alpha U_n$$

$$U_{n+1} = A_{n+1} e^{i\omega t}$$

$$\dot{U}_{n+1} = i\omega A_{n+1} e^{i\omega t}$$

$$\ddot{U}_{n+1} = -\omega^2 A_{n+1} e^{i\omega t}$$

$$U_n = A_n e^{i\omega t}$$

$$-\omega^2 m A_{n+1} e^{i\omega t} = -2\alpha A_{n+1} e^{i\omega t} + \alpha A_n e^{i\omega t}$$

Upon cancelling out terms involving $e^{i\omega t}$ we get

$$\boxed{-m\omega^2 A_{n+1} = -2\alpha A_{n+1} + \alpha A_n}$$

(b) Let us recast our results from Problem 1(a) as

$$2\alpha A_n - \alpha A_{n+1} - m\omega^2 A_n = 0$$

$$2\alpha A_{n+1} - \alpha A_n - m\omega^2 A_{n+1} = 0$$

or

$$\begin{pmatrix} 2\alpha - m\omega^2 & -\alpha \\ -\alpha & 2\alpha - m\omega^2 \end{pmatrix} \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or if we had placed the frequency terms on the right-hand side

$$\begin{pmatrix} 2\alpha & -\alpha \\ -\alpha & 2\alpha \end{pmatrix} \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix} = m\omega^2 \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix}$$

or

$$\underset{\sim}{A}\vec{v} = \lambda\vec{v}$$

where $\underset{\sim}{A} = \begin{pmatrix} 2\alpha & -\alpha \\ -\alpha & 2\alpha \end{pmatrix}$

$$\vec{v} = \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix}$$

$$\lambda = m\omega^2$$

(c) $A\vec{v} = \lambda\vec{v}$

$$\left|\underset{\sim}{A} - \underset{\sim}{I}\lambda\right| = 0$$

$$\begin{vmatrix} 2\alpha - m\omega^2 & -\alpha \\ -\alpha & 2\alpha - m\omega^2 \end{vmatrix} = 0$$

$$(2\alpha - m\omega^2)^2 - \alpha^2 = 0$$

$$m^2\omega^4 - 4\alpha m\omega^2 + 4\alpha^2 - \alpha^2 = 0$$

$$m^2\omega^4 - 4\alpha m\omega^2 + 3\alpha^2 = 0$$

$$\omega^2 = \frac{4\alpha m \pm \left[4\alpha^2 m^2\right]^{1/2}}{2m^2}$$

$$\omega^2 = \frac{4\alpha m \pm 2\alpha m}{2m^2}$$

$$\omega^2 = \frac{4\alpha m \pm 2\alpha m}{2m^2}$$

$$\omega_1^2 = \frac{2m\alpha}{2m^2} = \frac{\alpha}{m}$$

$$\boxed{\omega_1 = \sqrt{\frac{\alpha}{m}}}$$

$$\omega_2^2 = \frac{6\alpha m}{2m^2} = \frac{3\alpha}{m}$$

$$\boxed{\omega_2 = \sqrt{\frac{3\alpha}{m}}}$$

(d) What is $\vec{V}_1$ for ω_1?

$$\vec{V}_1 = \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix}$$

$$\underset{\sim}{A} \vec{V}_1 = \lambda_1 \vec{V}_1$$

$$\begin{pmatrix} 2\alpha & -\alpha \\ -\alpha & +2\alpha \end{pmatrix} \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix} = m\omega_1^2 \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix}$$

$$= m\frac{\alpha}{m} \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix}$$

or

$$\boxed{\begin{aligned} 2\alpha A_n - \alpha A_{n+1} &= \alpha A_n \\ -\alpha A_n + 2\alpha A_{n+1} &= \alpha A_{n+1} \end{aligned}}$$

Solving these equations we find

$$\alpha A_n = \alpha A_{n+1}$$

$$-\alpha A_{n+1} = -\alpha A_n$$

or

$$\boxed{A_n = A_{n+1}}$$

Let $A_n = a$, where a is an arbitrary constant

$$\vec{V}_1 = \begin{pmatrix} a \\ a \end{pmatrix}$$

What is $\vec{V}_2$ for ω_2?

$$\begin{pmatrix} 2\alpha & -\alpha \\ -\alpha & 2\alpha \end{pmatrix} \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix} = m\omega_2^2 \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix} = 3\alpha \begin{pmatrix} A_n \\ A_{n+1} \end{pmatrix}$$

or

$$\boxed{\begin{aligned} 2\alpha A_n - \alpha A_{n+1} &= 3\alpha A_n \\ -\alpha A_n + 2\alpha A_{n+1} &= 3\alpha A_{n+1} \end{aligned}}$$

Solving these equations we find

$$-\alpha A_n = +\alpha A_{n+1}$$

$$\alpha A_{n+1} = -\alpha A_n$$

$$\text{or} \quad A_n = -A_{n+1}$$

Let $A_2 = a$, where a is an arbitrary constant

$$\vec{V}_2 = \begin{pmatrix} a \\ -a \end{pmatrix}$$

(e)
$$\boxed{\begin{aligned} U_n &= b_1 e^{i\omega_1 t} + b_2 e^{i\omega_2 t} \\ U_{n+1} &= b_1 e^{i\omega_1 t} - b_2 e^{i\omega_2 t} \end{aligned}}$$

(f) Let $b_1 \neq 0$, $b_2 = 0$ so we have

$$u_n(t) = b_1 e^{i\omega_1 t}$$

$$U_{n+1}(t) = b_1 e^{i\omega_1 t}$$

If we introduce an arbitrary phase

$$b_1 = A_1 e^{-i\phi_1}$$

we have

$$U_n(t) = A_1 e^{i(\omega_1 t - \phi_1)}$$

$$U_{n+1}(t) = A_1 e^{i(\omega_1 t - \phi_1)}$$

and upon taking the real parts

$$U_n(t) = A_1 \cos(\omega_1 t - \phi_1)$$

$$U_{n+1}(t) = A_1 \cos(\omega_1 t - \phi_1)$$

which means that for this frequency ω_1 the two masses vibrate back and forth in unison.

(g) Let $b_1 = 0$, $b_2 \neq 0$ so we have from Part 1(e)

$$U_n(t) = b_2 e^{i\omega_2 t}$$

$$U_{n+1}(t) = -b_2 e^{i\omega_2 t}$$

In we introduce an arbitrary phase

$$b_2 = A_2 e^{-i\phi_2}$$

we have

$$U_n(t) = A_2 e^{i(\omega_2 t - \phi_2)}$$

$$U_{n+1}(t) = -A_2 e^{i(\omega_2 t - \phi_2)}$$

Upon taking the real parts we have

$$U_n(t) = A_2 \cos(\omega_2 t - \phi_2)$$

$$U_{n+1}(t) = -A_2 \cos(\omega_2 t - \phi_2)$$

Finally let us see what these two equations mean

Using our friend Euler again

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cos(\theta+\pi) = \frac{e^{i(\theta+\pi)} + e^{-i(\theta+\pi)}}{2}$$

$$\cos(\theta+\pi) = \frac{e^{i\pi}e^{i\theta} + e^{-i\pi}e^{-i\theta}}{2}$$

$$\cos(\theta+\pi) = -\left[\frac{e^{i\theta} + e^{-i\theta}}{2}\right]$$

$$\cos(\theta+\pi) = -\cos\theta$$

Thus

$$U_n(t) = A_2 \cos(\omega_2 t - \phi_2)$$

$$U_{n+1}(t) = -A_2 \cos(\omega_2 t - \phi_2)$$

$$U_{n+1}(t) = +A_2 \cos(\omega_2 t - \phi_2 + \pi)$$

and our two masses vibrate out of phase by 180°.

Problem 2

Diagonalize the following matrix Q

$$\underset{\sim}{Q} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\ (2-\lambda)(1-\lambda) \ + 0 - 2 \ + 2(2-\lambda) \ + 0 \ - 2(1-\lambda) = 0$$

Let us not express this as a third-order polynomial

$$(3-\lambda)\ (\lambda^2 - 3\lambda + 2) \ -2 \ + 4 - 2\lambda \ - 2 + 2\lambda = 0$$

$$(3-\lambda)\ (\lambda^2 - 3\lambda + 2) = 0$$

$$\lambda = 3 \Rightarrow (3-\lambda) = 0$$

$$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0$$
$$\Rightarrow \lambda = 1, 2$$

$$\left.\begin{array}{l}\lambda_1 = 1\\ \lambda_2 = 2\\ \lambda_3 = 3\end{array}\right\}$$ Note that our eigenvalues are distinct in this problem

$\lambda_1 = 1$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{array}{l} x - z = x \\ x + 2y + z = y \\ 2x + 2y + 3z = z \end{array} \Longrightarrow \begin{array}{l} z = 0 \\ x = -y \end{array}$$

$$\vec{v}_1 = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} \xrightarrow[\text{let } y = a]{} \vec{v}_1 = \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix}$$ when a is a constant

$\lambda_2 = 2$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{array}{l} x - z = 2x \\ x + 2y + z = 2y \\ 2x + 2y + 3z = 2z \end{array} \Longrightarrow -z = x$$

$$\Rightarrow 2x + 2y = -z = x$$

$$\Rightarrow x = -2y$$

Let $y = b$

$$\vec{V}_2 = \begin{pmatrix} -2b \\ b \\ 2b \end{pmatrix}, \text{ where } b \text{ is a constant}$$

$\lambda_3 = 3$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$x - z = 3x \quad \Rightarrow \quad 2x = -z$$
$$x + 2y + z = 3y \quad \Rightarrow \quad x + z = y$$
$$2x + 2y + 3z = 3z \quad \Rightarrow$$
$$2x + 2y = 0$$
$$x = -y$$

$$\vec{V}_3 = \begin{pmatrix} x \\ -x \\ -2x \end{pmatrix} \xrightarrow[\substack{\text{Let} \\ x = c}]{} \vec{V}_3 = \begin{pmatrix} c \\ -c \\ -2c \end{pmatrix}$$

Our eigenvectors $\vec{V}_1, \vec{V}_2, \vec{V}_3$ are

$$\vec{V}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{V}_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \quad \vec{V}_3 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

Where $a = b = c = 1$ for simplicity

$$\underset{\sim}{S} = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix}$$

Find $\underset{\sim}{S}^{-1}$ using Gauss-Jordan elimination

$$\left(\begin{array}{ccc|ccc} -1 & -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_1 \text{ and } r_2]{\text{interchange}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 + r_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-r_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_3/2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \end{array}\right)$$

$$\xrightarrow{-r_3 + r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow{-r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -\frac{1}{2} \end{array}\right)$$

$$\underset{\sim}{S}^{-1} = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & -1 & 0 \\ -1 & -1 & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$$

You can check that

$$\underset{\sim}{S}^{-1}\underset{\sim}{S} = \underset{\sim}{S}^{-1}\underset{\sim}{S} = \underset{\sim}{I}$$

$$\underset{\sim}{D} = \underset{\sim}{S}^{-1} \underset{\sim}{A} \underset{\sim}{S}$$

Here $\underset{\sim}{Q} = \underset{\sim}{A}$

$$\underset{\sim}{S}^{-1}\underset{\sim}{Q} = \frac{1}{2}\begin{pmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ -2 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 2-2 & 4-2 & 2-3 \\ -2-2 & -4 & 2-2 \\ -2-2-2 & -4-2 & 2-2-3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2 & -1 \\ -4 & -4 & 0 \\ -6 & -6 & -3 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1}\underset{\sim}{Q}\underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 0 & 2 & -1 \\ -4 & -4 & 0 \\ -6 & -6 & -3 \end{pmatrix}\begin{pmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1} Q \underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 2 & 2-2 & -2+2 \\ 4-4 & 8-4 & -4+4 \\ 6-6 & 12-6-6 & -6+6+6 \end{pmatrix}$$

$$\underset{\sim}{D} = \underset{\sim}{S}^{-1}\underset{\sim}{Q}\,\underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\underset{\sim}{D} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

It is interesting to note that in this problem for distinct eigenvalues, we see that our eigenvectors are linearly independent. This is not an accident but we will not prove this here!!!

Under what circumstances is the following true?

$$c_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -c_1 - 2c_2 + c_3 &= 0 \\ c_1 + c_2 - c_3 &= 0 \\ 2c_2 - 2c_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_2 &= c_3 \\ c_1 &= 0 \end{aligned}$$

and $-2c_2 + c_2 = 0$

$\Rightarrow c_2 = 0$

Thus c_1, c_2, c_3 all vanish and our vectors are linearly independent!

Problem 3

Diagonalize the following matrix Z:

$$Z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^3 - (1-\lambda) = 0$$

Rather than factor this out to obtain a cubic equation, let us find

$$(1-\lambda)\left[(1-\lambda)^2 - 1\right] = 0$$

or

$$(1-\lambda)\left[\lambda^2 - 2\lambda + 1 - 1\right] = 0$$

or

$$(1-\lambda)\,\lambda(\lambda-2) = 0$$

Thus $\lambda_1 = 0$

$\lambda_2 = 1$

$\lambda_3 = 2$

For $\lambda_1 = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow$$

$$\begin{aligned} x + z &= 0 \\ y &= 0 \\ x + z &= 0 \end{aligned} \Longrightarrow \begin{aligned} x &= -z \\ &\text{or} \\ z &= -x \end{aligned}$$

$$\vec{V}_1 = \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix}$$ Let $x = a$, an arbitrary constant

$$\vec{V}_1 = \begin{pmatrix} a \\ 0 \\ -a \end{pmatrix}$$

For $\lambda_2 = 1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longrightarrow \begin{aligned} x + z &= x \\ y &= y \\ x + z &= z \end{aligned}$$

Thus $z = 0$

$x = 0$

$y = b$, an arbitrary constant

$$\vec{V}_2 = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$$

For $\lambda_3 = 2$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x + z = 2z$$

$$y = 2y \qquad \Longrightarrow \qquad y = 0$$

$$x + z = 2z \qquad\qquad x = z = c,$$

an arbitrary constant

$$\vec{V}_3 = \begin{pmatrix} c \\ 0 \\ c \end{pmatrix}$$

Thus our eigenvectors are

$$\vec{V}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{V}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Where $a = b = c = 1$.

Are our eigenvectors linearly independent?

Under what circumstances is the following true?

$$c_1 \vec{V}_1 + c_2 \vec{V}_2 + c_3 \vec{V}_3 = \vec{0}\ ?$$

$$c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_1 + c_3 = 0$$
$$c_2 = 0$$
$$-c_1 + c_3 = 0$$

Now $c_2 = 0$ but

$2c_3 = 0 \Rightarrow c_3 = 0$

and $c_1 = 0$

so $\vec{V}_1, \vec{V}_2, \vec{V}_3$ are linearly independent

Let us construct

$$\underset{\sim}{S} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

and find $\underset{\sim}{S}^{-1}$ using Gauss-Jordon elimination

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_1 + r_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{r_3 / 2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right) \xrightarrow{-r_3 + r_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 1/2 \end{array}\right)$$

$$\underset{\sim}{S}^{-1} = \begin{pmatrix} +\tfrac{1}{2} & 0 & -\tfrac{1}{2} \\ 0 & 1 & 0 \\ \tfrac{1}{2} & 0 & \tfrac{1}{2} \end{pmatrix}$$

Let us check this

$$\underset{\sim}{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1}\underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1}\underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 1+1 & 0 & 1-1 \\ 0 & 2 & 0 \\ 1-1 & 0 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \underset{\sim}{I}$$

$$\underset{\sim}{S}\,\underset{\sim}{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 & -1+1 \\ 0 & 2 & 0 \\ -1+1 & 0 & 1+1 \end{pmatrix} = \underset{\sim}{I}$$

Let us diagonalize $\underset{\sim}{Z}$

In general, $\underset{\sim}{D} = \underset{\sim}{S}^{-1} \underset{\sim}{A} \underset{\sim}{S}$ or

$$\underset{\sim}{D} = \underset{\sim}{S}^{-1} \underset{\sim}{Z} \underset{\sim}{S}$$

$$\underset{\sim}{Z}\underset{\sim}{S} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{Z}\underset{\sim}{S} = \begin{pmatrix} 1-1 & 0 & 2 \\ 0 & 1 & 0 \\ 1-1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1} \underset{\sim}{Z} \underset{\sim}{S} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \underset{\sim}{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Problem 4

Diagonalize the following matrix $\underset{\sim}{M}$

$$\underset{\sim}{M} = \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix}$$

$$\underset{\sim}{M} = \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix}$$

$$|\underset{\sim}{M} - \lambda \underset{\sim}{I}| = \begin{vmatrix} 2-\lambda & 5 \\ 1 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) - 5 = 0$$

$$-5 + 12 - \lambda 6 - 2\lambda + \lambda^2 = 0$$

$$\lambda^2 - 8\lambda + 7 = 0$$

$$(\lambda - 7)(\lambda - 1) = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 7$$

$\lambda_1 = 1$

$$\begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Longrightarrow \quad \begin{matrix} 2x + 5y = x \\ x + 6y = y \end{matrix} \quad \longrightarrow \quad \begin{matrix} x = -5y \\ x = -5y \end{matrix}$$

$$\vec{v}_1 = \begin{pmatrix} -5y \\ y \end{pmatrix} \xrightarrow[\substack{y = a, \\ \text{an} \\ \text{arbitrary} \\ \text{constant}}]{\text{let}} \vec{v}_1 = \begin{pmatrix} -5a \\ a \end{pmatrix}$$

$\lambda_2 = 7$

$$\begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 7 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad \begin{matrix} 2x + 5y = 7x \\ x + 6y = 7y \end{matrix} \quad \Longrightarrow \quad \begin{matrix} 5y = 5x \\ x = y \end{matrix}$$

$$\vec{v}_2 = \begin{pmatrix} y \\ y \end{pmatrix} \xrightarrow[\substack{y = b \\ \text{be an} \\ \text{arbitrary} \\ \text{constant}}]{\text{let}} \vec{v}_2 = \begin{pmatrix} b \\ b \end{pmatrix}$$

Let $a = b = 1$ so our eigenvectors are

$$\vec{V}_1 = \begin{pmatrix} -5 \\ 1 \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let us build

$$\underset{\sim}{S} = \begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1} = \frac{1}{-5-1} \begin{pmatrix} 1 & -1 \\ -1 & -5 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 1 & -1 \\ -1 & -5 \end{pmatrix}$$

Check $\underset{\sim}{S}^{-1} \underset{\sim}{S} = \underset{\sim}{S} \underset{\sim}{S}^{-1} = \underset{\sim}{I}$ (It is easy to do!)

Diagonalize $\underset{\sim}{M}$

$$\underset{\sim}{D} = \underset{\sim}{S}^{-1} \underset{\sim}{A} \underset{\sim}{S} = \underset{\sim}{S}^{-1} \underset{\sim}{M} \underset{\sim}{S}$$

$$\underset{\sim}{D} = \underset{\sim}{S}^{-1} \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix} = \underset{\sim}{S}^{-1} \begin{pmatrix} -5 & 7 \\ 1 & 7 \end{pmatrix}$$

$$\underset{\sim}{D} = -\frac{1}{6} \begin{pmatrix} 1 & -1 \\ -1 & -5 \end{pmatrix} \begin{pmatrix} -5 & 7 \\ 1 & 7 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} -6 & 0 \\ 0 & -42 \end{pmatrix}$$

$$\underset{\sim}{D} = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

You can check this but because the eigenvalues are distinct it looks like the eigenvectors are linearly independent.

While we have not proven this it looks like a plausible trend!

Problem 5

Diagonalize the following matrix $\underset{\sim}{B}$

$$\underset{\sim}{B} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

For $\lambda_1 = 3$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} 2x + y &= 3x \\ x + 2y &= 3y \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= x \\ x &= y \end{aligned}$$

Let $x = y = a$, an arbitrary constant

$$\vec{V}_1 = \begin{pmatrix} a \\ a \end{pmatrix}$$

For $\lambda_2 = 1$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$2x + y = x \quad \Rightarrow \quad x = -y$$
$$x + 2y = y \quad \Rightarrow \quad x = -y$$

$$\vec{V}_2 = \begin{pmatrix} -b \\ b \end{pmatrix}$$ where b is an arbitrary constant

Thus our eigenvectors are

$$\vec{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

where we have set $a = b = 1$

Let us build $\underset{\sim}{S}$

$$\underset{\sim}{S} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1} = \frac{1}{1-(-1)} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Let us verify

$$\underset{\sim}{S}\,\underset{\sim}{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} =$$

$$\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \underset{\sim}{I}$$

$$\underset{\sim}{S}^{-1}\underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \underset{\sim}{I}$$

Thus $\underset{\sim}{D} = \underset{\sim}{S}^{-1}\underset{\sim}{A}\,\underset{\sim}{S} = \underset{\sim}{S}^{-1}\underset{\sim}{B}\,\underset{\sim}{S}$ for this problem

$$\underset{\sim}{B}\,\underset{\sim}{S} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2+1 \\ 3 & -1+2 \end{pmatrix}$$

$$\underset{\sim}{B}\,\underset{\sim}{S} = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1}\underset{\sim}{B}\underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1}\underset{\sim}{B}\underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 6 & -1+1 \\ -3+3 & 2 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1}\underset{\sim}{B}\underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1}\underset{\sim}{B}\underset{\sim}{S} = \frac{1}{2}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1}\underset{\sim}{B}\underset{\sim}{S} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Again these eigenvectors are linearly independent because

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} c_1 - c_2 = 0 \\ c_1 + c_2 = 0 \end{matrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \end{matrix}$$

and again our trend is in play as our eigenvalues are distinct!

Problem 6

Diagonalize the following matrix $\underset{\sim}{F}$

$$\underset{\sim}{F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$|\underset{\sim}{F} - \lambda \underset{\sim}{I}| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 (1-\lambda) = 0$$

$$\lambda = 1, 0, 0$$

$\lambda_1 = 1$
$\lambda_2 = 0$
$\lambda_3 = 0$

Note here that the eigenvalue are not distinct and two of them are degenerate

We will return to this later!

$\lambda_1 = 1$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$0x = 0 \Rightarrow x = 0$$
$$0y = 0 \Rightarrow y = 0$$
$$3x + z = z \Rightarrow z = z$$

Let $z = c$, an arbitrary constant

$$\vec{V}_1 = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$

$\lambda = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$0x = 0 \Rightarrow$ x is an arbitrary constant a

$0y = 0 \Rightarrow$ y is an arbitrary constant b

$3x + z = 0 \Rightarrow z = -3a$

$$\begin{pmatrix} a \\ b \\ -3a \end{pmatrix} \begin{cases} \xrightarrow[b=1]{\text{If } a=0} & \vec{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \xrightarrow[b=0]{\text{If } a=1} & \vec{V}_3 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \end{cases}$$

Thus our three eigenvectors are

$$\vec{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{V}_3 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

where we have set $c = 1$

Let us build $\underset{\sim}{S}$

$$\underset{\sim}{S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{pmatrix}$$

and find

$\underset{\sim}{S}^{-1}$ using Gauss-Jordan elimination

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3 \text{ and } r_1]{\text{Interchange}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{+3r_3 + r_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{S}^{-1} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Check

$$\underset{\sim}{S}^{-1} \underset{\sim}{S} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

$$\underset{\sim}{S}^{-1} \underset{\sim}{S} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

For $\underset{\sim}{F} = \underset{\sim}{A}$

$$\underset{\sim}{D} = \underset{\sim}{S}^{-1} \underset{\sim}{A} \underset{\sim}{S} = \underset{\sim}{S}^{-1} \underset{\sim}{F} \underset{\sim}{S}$$

$$\underset{\sim}{F} \underset{\sim}{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1} \underset{\sim}{F} \underset{\sim}{S} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underset{\sim}{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Finally are the eigenvectors linearly independent?

Under what circumstances is
the following true?

$$c_1\vec{V}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$c_1\begin{pmatrix}0\\0\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\0\\-3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{aligned} c_3 &= 0 \\ c_2 &= 0 \\ c_1 - c_3 &= 0 \end{aligned} \quad\Longrightarrow\quad c_1 = c_2 = c_3 = 0$$

Thus this is an example
where the eigenvalues are <u>not</u>
distinct but the matrix
is diagonalizable and the
eigenvectors are
linearly independent.
Contrast this with
Problem 2!

Problem 7

Diagonalize the following matrix $\underset{\sim}{K}$

$$\underset{\sim}{K} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 = 0$$

$$\left.\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 2 \end{aligned}\right\}$$ eigenvalues are not distint

For $\lambda_1 = 2$

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} 2x &= 2x \\ x + 2y &= 2y \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= x \\ x &= 0 \end{aligned}$$

$$\vec{V}_1 = \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

where a is an arbitrary constant

Same for $\lambda_2 = 2$

$$\vec{V}_2 = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Set $a = b = 1$

$$\vec{V}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underset{\sim}{S} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$\underset{\sim}{S}^{-1}$ does not exist as $|\underset{\sim}{S}| = 0$

Thus it seems that $\vec{V}_1$ and $\vec{V}_2$ are not linearly independent as the eigenvalues are not distinct!

Interesting!

Problem 8

Diagonalize the following matrix $\underset{\sim}{J}$

$$\underset{\sim}{J} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 (3-\lambda) = 0$$

$$\left.\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 2 \\ \lambda_3 &= 3 \end{aligned}\right\} \text{ eigenvalues are not distinct}$$

For $\lambda_1 = 2$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\begin{aligned} 3x &= 2x \\ 2y &= 2y \\ y + 2z &= 2z \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 0 \\ y &= 0 \\ z &= a \end{aligned} \; [\, a \text{ is an arbitrary constant} \,]$$

$$\vec{V}_1 = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$$

For $\lambda_2 = 2$

$$\vec{V}_2 = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$$

For $\lambda_3 = 3$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$\begin{array}{l} 3x = 3x \\ 2y = 3y \\ y + 2z = 3z \end{array} \Rightarrow \begin{array}{l} x = x \\ y = 0 \\ y = z \end{array} \Rightarrow z = y = 0$$

Let $x = b$, an arbitrary constant

$$\vec{V}_3 = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}$$

Our eigenvectors are

$$\vec{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{V}_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where $a = b = 1$

Are $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ linearly independent?

Under what circumstances is the following true?

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$c_1\begin{pmatrix}0\\0\\1\end{pmatrix} + c_2\begin{pmatrix}0\\0\\1\end{pmatrix} + c_3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{aligned} c_3 &= 0 \\ c_1 + c_2 &= 0 \\ 0c_1 + 0c_2 + 0c_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_3 &= 0 \\ c_1 &= -c_2 \end{aligned}$$

Therefore c_1 can be anything besides zero

Our vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly dependent

Well, $\underset{\sim}{S}^{-1} = ?$

$$\underset{\sim}{S} = \begin{pmatrix}0&0&1\\0&0&0\\1&1&0\end{pmatrix}$$

$$|\underset{\sim}{S}| = 0 + 0 + 0 - 0 - 0 - 0 = 0$$

$\underset{\sim}{S}^{-1}$ does not exist

so $\underset{\sim}{J}$ is not diagonalizable!

You see the trend here?

Problem 9

Diagonalize the following matrix $\underset{\sim}{A}$

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2(1-\lambda) + 2(1)(-1) + 2\cdot 0$$

$$+ 2(2-\lambda) - 0 - 2(1-\lambda) = 0$$

$$(2-\lambda)^2(1-\lambda) + 2 + 0 - 0 + 4 - 2\lambda$$
$$-2 + 2\lambda = 0$$

$$(2-\lambda)^2(1-\lambda) = 0$$

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

For $\lambda_1 = 1$

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x + 2y + 2z = x \quad \Rightarrow \quad y = -z$$

$$2y + z = y \qquad\qquad y = -z$$

$$-x + 2y + 2z = z \quad \Rightarrow \quad -x + 2y + z = 0$$

$$-x - 2z + z = 0$$

$$-x - z = 0$$

$$x = -z$$

Let $z = a$

$$\vec{V}_1 = \begin{pmatrix} -a \\ -a \\ a \end{pmatrix}$$

For $\lambda_2 = 2$

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$x = (y + z)\,2$$

$$x + 2y + 2z = 2x \quad \Rightarrow \quad y = -z$$

$$2y + z = 2y \quad \Rightarrow \quad z = 0$$

$$-x + 2y + 2z = 2z \qquad x = 2y$$

$$z = 0 \;;\; x = 2y$$

Let $y = b$

$$\vec{V}_2 = \begin{pmatrix} 2b \\ b \\ 0 \end{pmatrix}$$, where b is an arbitrary constant

Also

$$\vec{V}_3 = \begin{pmatrix} 2b \\ b \\ 0 \end{pmatrix}$$

Let $a = b = 1$

$$\vec{V}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \quad \vec{V}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \vec{V}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Are these vectors linearly independent?

Under what conditions is the following true?

$$C_1 \vec{V}_1 + C_2 \vec{V}_2 + C_3 \vec{V}_3 = \vec{0}$$

$$C_1 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -C_1 + 2C_2 + 2C_3 &= 0 \\ -C_1 + C_2 + C_3 &= 0 \\ C_1 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} C_1 &= 0 \\ C_2 &= -C_3 \end{aligned}$$

C_2 can be anything

Our eigenvectors $\vec{V}_1, \vec{V}_2, \vec{V}_3$ are not linearly independent!

What is $\underset{\sim}{S}^{-1}$?

$$\underset{\sim}{S} = \begin{pmatrix} -1 & 2 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{S}^{-1} = ??$$

$$|\underset{\sim}{S}| = (-1)(1)(0) + 2(1)(1)$$

$$-2 \cdot 0 - 2 - 0 - 0 = 0$$

Since $|\underset{\sim}{S}| = 0$, $\underset{\sim}{S}^{-1}$ does not exist!

By now you should be getting a clearer idea that all matrices are not diagonalizable and under what conditions this is true!