AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021
Zoom Lecture: Tu: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET X
(due Tuesday, July 27, 2021)

Problem 1

Suppose we have a new basis $\vec{\boldsymbol{B}}'$ where the basis vectors are given by

$$\hat{e}'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{e}'_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{e}'_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and suppose we have a vector

$$\vec{\boldsymbol{v}}_{\vec{\boldsymbol{B}}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

where of course $\vec{\boldsymbol{B}}$ is the standard basis. Find the change in basis matrix for the problem and find $\vec{\boldsymbol{v}}_{\vec{\boldsymbol{B}}'}$. Explicitly check that the inverse of the change in basis matrix does what it is supposed to do correctly.

Problem 2

Suppose we have a new basis $\vec{B}'$ where the basis vectors are given by

$$\hat{e}_1' = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\hat{e}_2' = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

and suppose we have a vector

$$\vec{v}_B = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

which is expressed in the standard basis. Find the change in basis matrix that will express $\vec{v}_{\vec{B}}$ in terms of your new basis. Explicitly check that the inverse of the change in basis matrix does what it is supposed to do correctly.

Problem 3

Find the eigenvalues and eigenvectors of the following matrix $\mathbf{Z}$:

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1 + \lambda_2 + \lambda_3 = Tr}\,\mathbf{Z}$ and $\mathbf{det\ Z} = \boldsymbol{\lambda_1\lambda_2\lambda_3}$ where λ_1, λ_2, and λ_3 are the eigenvalues of $\mathbf{Z}$. Recall from Problem Set III that Tr $\mathbf{Z}$ is the trace of the matrix $\mathbf{Z}$ which is the sum of all the diagonal elements of a matrix.

Problem 4

Find the eigenvalues and eigenvectors of the following matrix M:

$$\mathrm{M} = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1 + \lambda_2 = Tr}\,\mathrm{M}$ and $\mathbf{det}\ \mathrm{M} = \boldsymbol{\lambda_1\lambda_2}$ where λ_1 and λ_2 are the eigenvalues of M.

Problem 5

Find the eigenvalues and eigenvectors of the following matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{pmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{3} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} = \boldsymbol{Tr}\,\mathbf{P}$ and $\mathbf{det}\ \mathbf{P} = \boldsymbol{\lambda_1 \lambda_2}$ where λ_1 and λ_2 are the eigenvalues of $\mathbf{P}$.

Problem 6

Find the eigenvalues and eigenvectors of the following matrix $\mathbf{Y}$:

$$\mathbf{Y} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{-2} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} + \boldsymbol{\lambda_3} = \boldsymbol{Tr}\,\mathbf{Y}$ and $\mathbf{det}\ \mathbf{Y} = \boldsymbol{\lambda_1 \lambda_2 \lambda_3}$ where λ_1, λ_2, and λ_3 are the eigenvalues of $\mathbf{Y}$.

Problem 7

Find the eigenvalues and eigenvectors of the following matrix $\mathbf{D}$:

$$\mathbf{D} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} = \boldsymbol{Tr}\,\mathbf{D}$ and $\mathbf{det}\ \mathbf{D} = \boldsymbol{\lambda_1 \lambda_2}$ where λ_1 and λ_2 are the eigenvalues of $\mathbf{D}$.

Problem 8

Find the eigenvalues and eigenvectors of the following matrix $\mathbf{T}$:

$$\mathbf{T} = \begin{pmatrix} \mathbf{3} & \mathbf{-2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-4} & \mathbf{-3} \\ \mathbf{2} & \mathbf{-4} & \mathbf{0} \end{pmatrix}$$

How many eigenvectors are linearly independent? Also show explicitly that $\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2} + \boldsymbol{\lambda_3} = \boldsymbol{Tr}\,\mathbf{T}$ and $\mathbf{det}\ \mathbf{T} = \boldsymbol{\lambda_1 \lambda_2 \lambda_3}$ where λ_1, λ_2, and λ_3 are the eigenvalues of $\mathbf{T}$.

Problem 9

Please look at the absolutely beautiful animated lecture by Grant Sanderson "Change of Basis, Essence of Linear Algebra: Chapter 13 (12:50 minutes)." His geometric discussion of the basic concepts in linear algebra is so spectacular and I could never accomplish what he does in a lecture. Learning in the 21^{th} century is accomplished through all sorts of mechanisms (e.g. lectures, problem sets, reading, recitation sections, videos, etc.) so we should take advantage of all of these approaches!

In particular, in this video he shows how the change of basis matrix works in a very clear and geometric way! I highly suggest looking at it once you have mastered Lecture 10 and Problem Set 10.

Problem 10

Please look at the absolutely beautiful animated lecture by Grant Sanderson "Eigenvectors and Eigenvalues, Essence of Linear Algebra: Chapter 14 (just look at the first 13:04 minutes)." His geometric discussion of the basic concepts in linear algebra is so spectacular and I could never accomplish what he does in a lecture. Learning in the 21^{th} century is accomplished through all sorts of mechanisms (e.g. lectures, problem sets, reading, recitation sections, videos, etc.) so we should take advantage of all of these approaches!

In particular, in this video he describes eigenvectors and eigenvalues in a very clear and geometric way! His discussion of the determinant using geometry is interesting. You can either skip it, as it is very brief, or look at his Lecture 6 if you want. I highly suggest looking at his Lecture 14 once you have mastered our Lecture 10 and Problem Set 10.

Python Exercise 10

1. The following is a Python script (a fancy word for a program) for finding the eigenvalues and eigenvectors of a matrix

```
import numpy as np

A = np.array([[1,2],[2,1]])

print(A)

L,V =np.linalg.eig(A)# Here L stands for the eigenvalues (Lambda) and V stands for the eigenvectors (Vectors)

print(L)

print(V)
```

Note that the function **np.linalg.eig()** returns the eigenvalues in a vector and the eigenvectors in a matrix. Also note that the eigenvectors are normalized. Use this script to verify your results in Problems 3-8 of this problem set.

2. Create the following random matrices of sizes: **(5 x 5)**, **(10 x 10)**, **(20 x 20)**, and **(50 x 50)**. Use Python to find their eigenvalues and eigenvectors. See if your matrices are invertible.

3. The n^{th} power of a square matrix is denoted by A^n . Why won't this operation work for a rectangular matrix? In Python this power is formed using the function **np.linalg.matrix_power(A,2)** where n = 2, for example.

```
import numpy as np

A = np.array([[1,5],[3,4]])

print(A)

T =np.linalg.matrix_power(A,2)# evaluate the matrix A raised to the power of 2

print(A)
```

Experiment with this Python script for matrices where the matrix elements are all larger than 1 and all less than one. Do you see any trends? We will come back to this idea in Problem Set XII.

Problem 1

$$\vec{V}_{\vec{B}} = \underset{\sim}{P}\ \vec{V}_{\vec{B}'} \qquad \text{Change of basis formula}$$

$$\vec{B} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$$

$$\vec{B}' = \{\hat{e}_1', \hat{e}_2', \hat{e}_3'\}$$

$$\hat{e}_1' = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \hat{e}_2' = \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \hat{e}_3' = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\underset{\sim}{P} = (\hat{e}_1'\ \hat{e}_2'\ \hat{e}_3') = \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

$$\underset{\sim}{P} = \begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$$

Find $\underset{\sim}{P}^{-1}$ using Gauss-Jordan elimination

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&0&1&0\\0&0&1&0&0&1\end{array}\right) \xrightarrow{-r_3 + r_2} \left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&0&1&-1\\0&0&1&0&0&1\end{array}\right)$$

$$\xrightarrow{-r_3 + r_1} \left(\begin{array}{ccc|ccc}1&1&0&1&0&-1\\0&1&0&0&1&-1\\0&0&1&0&0&1\end{array}\right) \xrightarrow{-r_2 + r_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\sim}{P}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Let us check this

$$\underset{\sim}{P}\,\underset{\sim}{P}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{P}\,\underset{\sim}{P}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

$$\underset{\sim}{P}^{-1}\underset{\sim}{P} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

$$\vec{V}_{\vec{B}'} = \underset{\sim}{P}^{-1} \vec{V}_{\vec{B}}$$

$$V_{\vec{B}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \underset{\sim}{P}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{V}_{\vec{B}'} = \underset{\sim}{P}^{-1} \vec{V}_{\vec{B}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 2-3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

$$\vec{V}_{\vec{B}'} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}_{\vec{B}'}$$

Let us check this

$$\vec{V}_{\vec{B}'} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}_{\vec{B}'} = -1 \, \hat{e}_1' - \hat{e}_2' + 3 \, \hat{e}_3'$$

$$= -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 -1 +3 \\ -1 + 3 \\ 1 \cdot 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \vec{V}_{\vec{B}}$$

Thus everything checks out !!!

Problem 2

$$\vec{V}_{\vec{B}} = \underset{\sim}{P}\, \vec{V}_{\vec{B}'} \qquad \text{Change in basis formula}$$

$$\vec{B}' = \{ \hat{e}_1', \hat{e}_2' \} \qquad \vec{B} = \{ \hat{e}_1, \hat{e}_2 \}$$

$$\hat{e}_1' = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \hat{e}_2' = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

$$\vec{V}_{\vec{B}} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\vec{V}_{\vec{B}'} = \underset{\sim}{P}^{-1}\, \vec{V}_{\vec{B}}$$

Let us find $\underset{\sim}{P}$

$$\underset{\sim}{P} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$

Now what is $\underset{\sim}{P}^{-1}$?

$$\underset{\sim}{P}^{-1} = \frac{1}{4\cdot 2 - (3)(1)} \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$$

$$\underset{\sim}{P}^{-1} = \frac{1}{8+3} \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{pmatrix}$$

$$\underset{\sim}{P}\,\underset{\sim}{P}^{-1} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

$$\vec{V}_{\vec{B}'} = \underset{\sim}{P}^{-1}\vec{V}_{\vec{B}}$$

$$\vec{V}_{\vec{B}'} = \begin{pmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\vec{V}_{\vec{B}'} = \begin{pmatrix} 16/11 + 18/11 \\ -4/11 + 12/11 \end{pmatrix} = \begin{pmatrix} 34/11 \\ 8/11 \end{pmatrix}$$

Check

$$\vec{V}_{\vec{B}} = \underset{\sim}{P}\,\vec{V}_{\vec{B}'} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 34/11 \\ 8/11 \end{pmatrix}$$

$$\vec{V}_{\vec{B}} = \begin{pmatrix} 68/11 - 24/11 \\ 34/11 + 32/11 \end{pmatrix} = \begin{pmatrix} 44/11 \\ 66/11 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

✓

Also

$$\vec{V}_{\vec{B}'} = \begin{pmatrix} 34/11 \\ 8/11 \end{pmatrix}_{\vec{B}'} = 34/11\ \tilde{e}'_1 + 8/11\ \tilde{e}'_2$$

$$\vec{V}_{\vec{B}'} = \frac{34}{11}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 8/11\begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

$$\vec{V}_{\vec{B}'} = \begin{pmatrix} 68/11 \\ 34/11 \end{pmatrix} + \begin{pmatrix} -24/11 \\ 32/11 \end{pmatrix} = \begin{pmatrix} 44/11 \\ 66/11 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$= \vec{V}_{\vec{B}}$$

cool!!

Problem 3

$$\underset{\sim}{Z} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$|\underset{\sim}{Z} - \lambda \underset{\sim}{I}| = \begin{vmatrix} (1-\lambda) & 0 & 1 \\ 0 & (1-\lambda) & 0 \\ 1 & 0 & (1-\lambda) \end{vmatrix} = 0$$

$$(1-\lambda)^3 + 0\cdot 0\cdot 1 + 1\cdot 0\cdot 0$$

$$- (1-\lambda) - 0\cdot 0\,(1-\lambda) - (1-\lambda)0\cdot 0$$

$$= 0$$

$$(1-\lambda)^3 - (1-\lambda) = 0$$

Clearly $\lambda_1 = 1$ solves this equation

Let us simplify the equation to get

$$(1-\lambda)^2 = 1$$

or

$$\lambda^2 - 2\lambda + 1 = 1$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

The other two roots are $\lambda_2 = 0$

$\lambda_3 = 2$

For $\lambda_1 = 1$ $\quad\quad \vec{V}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$Z\vec{V}_1 = \lambda_1 \vec{V}_1$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x + z = x$$
$$y = y$$
$$x + z = z$$

Let $y = a$, a constant

Clearly $x = 0$ and $z = 0$

$$\vec{V}_1 = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}$$

For $\lambda_2 = 0$ $\qquad$ $\vec{V}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\underset{\sim}{Z}\,\vec{V}_2 = \lambda_2 \vec{V}_2$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x + z &= 0 \\ y &= 0 \\ x + z &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -z \\ y &= 0 \end{aligned}$$

Let $z = b$

$$\vec{V}_2 = \begin{pmatrix} -b \\ 0 \\ b \end{pmatrix}$$

For $\lambda_3 = 2$ $\qquad$ $\vec{V}_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\underset{\sim}{Z}\,\vec{V}_3 = \lambda_3 \vec{V}_3$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x + z = 2x$$
$$y = 2y$$
$$x + z = 2z$$

Clearly $y = 0$

$z = x$

$$\vec{v}_3 = \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} \qquad \text{Let } x = c$$

$$\vec{v}_3 = \begin{pmatrix} c \\ 0 \\ c \end{pmatrix}$$

If we choose $a = b = c = 1$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Are these eigenvectors linearly independent?
Under what circumstances is the following true?

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-c_2 + c_3 = 0$$

$$c_1 = 0$$

$$c_2 + c_3 = 0$$

Clearly $c_1 = 0$

$2c_3 = 0$ or $c_3 = 0$

and

$c_2 = 0$

so our three vectors are linearly independent

Finally for

$$\underset{\sim}{Z} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$Tr\, \underset{\sim}{Z} = 1 + 1 + 1 = 3$$

$$|\underset{\sim}{Z}| = 1 - 0 - 0 - 1 - 0 - 0 = 0$$

$$Tr\, \underset{\sim}{Z} = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 0 + 2 = 3$$

$$|\underset{\sim}{Z}| = \lambda_1 \lambda_2 \lambda_3 = 1(0)(2) = 0$$

<u>Problem 4</u>

$$\underset{\sim}{M} = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & 4 \\ -2 & 5-\lambda \end{vmatrix} = 0$$

$$-\left[(5-\lambda)(1+\lambda)\right] + 8 = 0$$

$$-\left[5 - \lambda^2 - \lambda + 5\lambda\right] + 8 = 0$$

$$-5 + \lambda^2 + \lambda - 5\lambda + 8 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 3$$

For λ_1

$$\underset{\sim}{M}\vec{v}_1 = \lambda_1 \vec{v}_1 \qquad \lambda_1 = 1$$

$$\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} -x + 4y &= x \\ -2x + 5y &= y \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x &= 4y \\ 2x &= 4y \end{aligned}$$

or

$$x = 2y$$

$$\vec{V}_1 = \begin{pmatrix} 2y \\ y \end{pmatrix}$$

Let $y = a$ where a is a scaler constant

$$\vec{V}_1 = \begin{pmatrix} 2a \\ a \end{pmatrix}$$

If we set $a = 1$

$$\vec{V}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For λ_2

$$\underset{\sim}{M}\,\vec{V}_2 = \lambda_2 \vec{V}_2$$

$$\lambda_2 = 3$$

$$\vec{V}_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} -x + 4y &= 3x \\ -2x + 5y &= 3y \end{aligned} \quad \Longrightarrow \quad \begin{aligned} 4y &= 4x \\ 2y &= 2x \end{aligned} \quad \text{or } y = x$$

$$\vec{V}_2 = \begin{pmatrix} x \\ x \end{pmatrix}$$

Let us set $x = b$, where b is a scaler

$$\vec{V}_2 = \begin{pmatrix} b \\ b \end{pmatrix} \xrightarrow[\text{For } b = 1]{} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Are these eigenvectors linearly independent?

To answer this question we ask under what circumstances is the following equation true?

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \tilde{0}$$

or

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 2c_1 + c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 0 \\ c_2 &= 0 \end{aligned}$$

Thus $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

Note that $\underset{\sim}{M} = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$

$$Tr\, \underset{\sim}{M} = -1 + 5 = 4$$

$$|\underset{\sim}{M}| = -5 + 8 = 3$$

$$Tr\, \underset{\sim}{M} = \lambda_1 + \lambda_2 = 1 + 3 = 4$$

$$|\underset{\sim}{M}| = 1 \cdot 3 = 3 = \lambda_1 \lambda_2$$

Problem 5

$$\underset{\sim}{P} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(1-\lambda) + 1 = 0$$

$$3 - \lambda - 3\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda_1 = 2 \quad ; \quad \lambda_2 = 2$$

These two eigenvalues are not distinct so we only expect one eigenvector for this problem!

For $\lambda = 2$ $\qquad \vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} x - y &= 2x \\ x + 3y &= 2y \end{aligned} \quad \Rightarrow \quad \begin{aligned} -y &= x \\ y &= -x \end{aligned}$$

$\vec{v}_1 = \begin{pmatrix} x \\ -x \end{pmatrix}$ Let us set $x = a$

$$\vec{v}_1 = \begin{pmatrix} a \\ -a \end{pmatrix} \quad \text{or for } a = 1$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Apparently there is only one linearly independent eigenvector for

$$\underset{\sim}{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

since the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 2$ are not distinct

Note that $\underset{\sim}{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

$$Tr\ \underset{\sim}{A} = 1 + 3 = 4$$

$$|\underset{\sim}{A}| = 3 + 1 = 4$$

$$Tr\ \underset{\sim}{A} = \lambda_1 + \lambda_2 = 2 + 2 = 4$$

$$|\underset{\sim}{A}| = \lambda_1 \lambda_2 = 2(2) = 4$$

Problem 6

$$\underset{\sim}{Y} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} (1-\lambda) & 1 & 1 \\ 1 & -\lambda & -2 \\ 1 & -1 & (1-\lambda) \end{vmatrix} = -(1-\lambda)^2\lambda \quad -2 \quad -1$$

$$\lambda - (1-\lambda) \quad - 2(1-\lambda) = 0$$

$$-\lambda(1-\lambda)^2 \quad -3 + \lambda \quad -3(1-\lambda) = 0$$

$$-\lambda(1-\lambda)^2 \quad -3 + \lambda - 3 + 3\lambda = 0$$

$$-\lambda(1-\lambda)^2 \quad -6 + 4\lambda = 0$$

$$-\lambda\left[1 - 2\lambda + \lambda^2\right] - 6 + 4\lambda = 0$$

$$-\lambda + 2\lambda^2 - \lambda^3 - 6 + 4\lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + 3\lambda - 6 = 0$$

$$\lambda^3 - 2\lambda^2 - 3\lambda + 6 = 0$$

You can consult GOOGLE on ways to solve a cubic equation. For example, since the prefactor of λ^3 is 1, then the solution to the bracketed term below

$$\left[\lambda^3 - 2\lambda^2 - 3\lambda\right] + 6 = 0$$

is some factor of 6.

For $\lambda = 1$ $\quad 1 - 2 - 3 + 6 \neq 0$

$\lambda = 2$ $\quad 8 - 8 - 6 + 6 = 0$

Thus $\lambda_1 = 2$ is one root of our cubic equation so we have

$$(\lambda - 2)\,(\lambda^2 + a\lambda + b) = 0$$

where we need to find a, b by synthetic division

First, write down the coefficients of the original equation on the top row of a table, with a dividing line and then the known root on the right

$$\begin{array}{cccc|c} 1 & -2 & -3 & +6 & \lambda = 2 \\ \hline & & & & \end{array}$$

Leave one spare row, and then add a horizontal line below it. First, take the first number (1) down to the row below your horizontal line

$$\begin{array}{cccc|c} 1 & -2 & -3 & 6 & \lambda = 2 \\ & & & & \\ \hline 1 & & & & \end{array}$$

Now multiply the number you have just brought down by the known root or in this case $1 \times 2 = 2$ and this is written below the next number in the list, as follows

$$\begin{array}{cccc|c} 1 & -2 & -3 & 6 & \lambda = 2 \\ & 2 & & & \\ \hline 1 & & & & \end{array}$$

Next add the numbers in the second column and put the result below the horizontal line

$$\begin{array}{cccc|c} 1 & -2 & -3 & 6 & \lambda = 2 \\ & 2 & & & \\ \hline 1 & 0 & & & \end{array}$$

Repeat this process with the new number below the horizontal line. Multiply by the root and put the answer in the empty space in the next column and then add the column to get a new number on the bottom row. This leaves

$$\begin{array}{cccc|c} 1 & -2 & -3 & 6 & \lambda = 2 \\ & 2 & 0 & & \\ \hline 1 & 0 & -3 & & \end{array}$$

Repeat the process again

1	-2	-3	6	$\lambda = 2$
	2	0	-6	
1	0	-3	0	

The last answer should be zero and you have found a valid root. If not you have made an error somewhere.

Thus $a = 0$ and $b = -3$ so the remaining roots are found from

$$\lambda^2 - 3 = 0$$

of the equation

$$(\lambda - 2)\ (\lambda^2 + a\lambda + b) = 0$$

and

$$\lambda_2 = +\sqrt{3}$$

$$\lambda_3 = -\sqrt{3}$$

In summary

$$\lambda_1 = 2$$

$$\lambda_2 = \sqrt{3}$$

$$\lambda_3 = -\sqrt{3}$$

Let us check these eigenvalues

$$\underset{\sim}{Y} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$|\underset{\sim}{Y}| = 1 \cdot 0 \cdot 1 + 1(-2)(1) + 1(1)(-1)$$

$$- (1)(0)(1) - (1)(1)(1) - (1)(-2)(-1)$$

$$= -2 - 1 - 1 - 2 = -6$$

$$\det \underset{\sim}{Y} = |\underset{\sim}{Y}| = \lambda_1 \lambda_2 \lambda_3 = 2\sqrt{3}(-\sqrt{3}) = -6$$

$$Tr\, \underset{\sim}{Y} = 1 + 0 + 1 = 2 = \lambda_1 + \lambda_2 + \lambda_3 = 2 + \sqrt{3} - \sqrt{3}$$
$$= 2$$

Now let us find our eigenvectors

$\lambda_1 = 2$

$$\underset{\sim}{Y} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \qquad \vec{v}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underset{\sim}{Y} \vec{v}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x + y + z = 2x$$
$$x - 2z = 2y$$
$$x - y + z = z2$$

or

$$-x + y + z = 0 \qquad (1)$$
$$x - 2y - 2z = 0 \qquad (2)$$
$$x - y - z = 0 \qquad (3)$$

If we add Eq. (1) and Eq. (2)

$$-y - z = 0 \quad \Rightarrow \quad \boxed{y = -z} \qquad (4)$$

Using (4) in Eq. (2)

$$x - 2(-z) - 2z = 0$$
$$x + 2z - 2z = 0 \quad \Rightarrow \quad x = 0$$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ a \\ -a \end{pmatrix} \xrightarrow[\text{where } a \text{ is a constant}]{\text{Let } a = 1} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\lambda_2 = \sqrt{3}$

$$\vec{V}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underset{\sim}{Y}\, \vec{V}_2 = \lambda_2 \vec{V}_2$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x + y + z = \sqrt{3}\, x$$
$$x - 2z = \sqrt{3}\, y$$
$$x - y + z = \sqrt{3}\, z$$

or

$$x(1-\sqrt{3}) + y + z = 0 \quad (1)$$
$$x - \sqrt{3}\, y - 2z = 0 \quad (2)$$
$$x - y + z(1-\sqrt{3})z = 0 \quad (3)$$

Subtracting (3) from (2)

$$y(1-\sqrt{3}) - z + \sqrt{3}\, z - 2z = 0$$
$$y(1-\sqrt{3}) + z(\sqrt{3} - 3) = 0$$

$$y = \frac{z\,(3-\sqrt{3})}{1-\sqrt{3}} \quad (4)$$

Plugging ④ into ① allows us to find an expression for $x = x(z)$

$$x(1-\sqrt{3}) + z\frac{(3-\sqrt{3})}{1-\sqrt{3}} + z = 0$$

$$(1-\sqrt{3})x + z\frac{[3-\sqrt{3}+1-\sqrt{3}]}{1-\sqrt{3}} = 0$$

$$(1-\sqrt{3})x + \frac{z[4-2\sqrt{3}]}{1-\sqrt{3}} = 0$$

$$x = \frac{-z[4-2\sqrt{3}]}{(1-\sqrt{3})^2}$$

$$x = \frac{-z[4-2\sqrt{3}]}{1+3-2\sqrt{3}} = -z$$

Thus

$$\vec{v}_2 = \begin{pmatrix} -z \\ z\frac{(3-\sqrt{3})}{1-\sqrt{3}} \\ z \end{pmatrix}$$

Let $z = a$
a constant

$$\vec{v}_2 = \begin{pmatrix} -a \\ a\frac{[3-\sqrt{3}]}{1-\sqrt{3}} \\ a \end{pmatrix} \xrightarrow[]{\text{let } a=1} \begin{pmatrix} -1 \\ \frac{3-\sqrt{3}}{1-\sqrt{3}} \\ 1 \end{pmatrix}$$

Finally, $\lambda_3 = -\sqrt{3}$

$$\vec{v}_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underset{\sim}{Y}\, \vec{v}_3 = \lambda_3 \vec{v}_3$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\sqrt{3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x + y + z = -\sqrt{3}\, x$$
$$x - 2z = -\sqrt{3}\, y$$
$$x - y + z = -\sqrt{3}\, z$$

or

$$x(1+\sqrt{3}) + y + z = 0 \qquad (1)$$
$$x + \sqrt{3}\, y - 2z = 0 \qquad (2)$$
$$x - y + z(1+\sqrt{3}) = 0 \qquad (3)$$

Let us find (2) − (3)

$$\sqrt{3}y + y - 2z - z(1+\sqrt{3}) = 0$$

$$y(1+\sqrt{3}) - 2z - z - z\sqrt{3} = 0$$

$$y(1+\sqrt{3}) - 3z - z\sqrt{3} = 0$$
$$y(1+\sqrt{3}) - z(3+\sqrt{3}) = 0$$

$$y = z\frac{(3+\sqrt{3})}{(1+\sqrt{3})} \quad ④$$

Substituting ④ into ① yields

$$x(1+\sqrt{3}) + \frac{z(3+\sqrt{3})}{(1+\sqrt{3})} + z = 0$$

$$x(1+\sqrt{3}) + z3 + z\sqrt{3} + z + z\sqrt{3} = 0$$

$$x(1+\sqrt{3}) + \frac{4z + 2z\sqrt{3}}{(1+\sqrt{3})} = 0$$

$$x(1+\sqrt{3}) + \frac{z(4+2\sqrt{3})}{(1+\sqrt{3})} = 0$$

$$x = -\frac{z(4+2\sqrt{3})}{(1+\sqrt{3})^2}$$

$$x = \frac{-z(4+2\sqrt{3})}{3+1+2\sqrt{3}} = -z$$

$$\vec{V}_3 = \begin{pmatrix} -z \\ z\frac{(3+\sqrt{3})}{(1+\sqrt{3})} \\ z \end{pmatrix}$$

Let $z = a$ and set $a = 1$

$$\vec{V}_3 = \begin{pmatrix} -1 \\ \frac{(3+\sqrt{3})}{(1+\sqrt{3})} \\ 1 \end{pmatrix}$$

which is same as $\vec{V}_2$ so

Thus

$$\vec{V}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} -1 \\ \dfrac{3-\sqrt{3}}{1-\sqrt{3}} \\ 1 \end{pmatrix}$$

$$\vec{V}_3 = \begin{pmatrix} -1 \\ \dfrac{3+\sqrt{3}}{1+\sqrt{3}} \\ 1 \end{pmatrix}$$

Are these three vectors linearly independent?

Under what circumstances is the following true?

$$C_1 \vec{V}_1 + C_2 \vec{V}_2 + C_3 \vec{V}_3 = \vec{0}$$

$$C_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ \dfrac{3-\sqrt{3}}{1-\sqrt{3}} \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} -1 \\ \dfrac{3+\sqrt{3}}{1+\sqrt{3}} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-C_2 - C_3 = 0 \qquad (1)$$

$$C_1 + C_2 \frac{(3-\sqrt{3})}{1-\sqrt{3}} + C_3 \frac{(3+\sqrt{3})}{1+\sqrt{3}} = 0 \qquad (2)$$

$$-C_1 + C_2 + C_3 = 0 \qquad (3)$$

From (1) $c_2 = -c_3$ which is only true if $c_2 = c_3 = 0$

Thus $c_1 = 0$

and all three vectors are linearly independent

Problem 7

$$\underset{\sim}{D} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 2 = 0$$

$$2 + \lambda^2 - 3\lambda - 2 = 0 \qquad \lambda^2 - 3\lambda = 0$$

$$\lambda(\lambda - 3) = 0$$

$$\lambda_1 = 0$$
$$\lambda_2 = 3$$

For $\lambda_1 = 0$

$$\underset{\sim}{D}\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} x + y &= 0 \\ 2x + 2y &= 0 \end{aligned} \Rightarrow -x = y$$

Let $x = a$ $\qquad \vec{v}_1 = \begin{pmatrix} a \\ -a \end{pmatrix}$

For $\lambda_2 = 3$

$$\underset{\sim}{D}\vec{V}_2 = \lambda_2 \vec{V}_2 \qquad \vec{V}_2 = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underset{\sim}{D}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

$$\begin{array}{l} x + y = 3x \\ 2x + 2y = 3y \end{array} \Rightarrow \qquad 2x = y$$

Let $x = b$

$$\vec{V}_2 = \begin{pmatrix} b \\ 2b \end{pmatrix}$$

$$\vec{V}_1 = \begin{pmatrix} a \\ -a \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} b \\ 2b \end{pmatrix}$$

Let $a = 1$, $b = 1$

$$\vec{V}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \vec{V}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Are $\vec{V}_1$ and $\vec{V}_2$ linearly independent?

Under what circumstances is

$$C_1\vec{V}_1 + C_2\vec{V}_2 = \vec{O}$$

$$C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 + c_2 = 0$$
$$\Rightarrow \quad 3c_2 = 0 \quad \Rightarrow \quad c_2 = c_1 = 0$$
$$-c_1 + 2c_2 = 0$$

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent

Also note that

$$\lambda_1 + \lambda_2 = 0 + 3 = 3$$

$$|\underset{\sim}{D}| = 2 - 2 = 0 \quad = \lambda_1 \lambda_2 = 0 \cdot 3 = 0$$

Problem 8

$$\underset{\sim}{T} = \begin{pmatrix} 3 & -2 & -1 \\ 3 & -4 & -3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$\begin{vmatrix} (3-\lambda) & -2 & -1 \\ 3 & (-4-\lambda) & -3 \\ 2 & -4 & -\lambda \end{vmatrix} =$$

$$\lambda(4+\lambda)(3-\lambda) + 12 + 12 - 2(4+\lambda) - 6\lambda$$

$$-12(3-\lambda) = 0$$

$$\lambda[12 + 3\lambda - 4\lambda - \lambda^2] + 24 - 8 - 2\lambda - 6\lambda$$

$$-36 + 12\lambda = 0$$

$$-\lambda^3 - 4\lambda^2 + 3\lambda^2 + 12\lambda + 16 - 8\lambda + 12\lambda - 36 = 0$$

$$\lambda^3 + 4\lambda^2 - 3\lambda^2 - 12\lambda - 16 + 8\lambda - 12\lambda + 36 = 0$$

$$\lambda^3 + \lambda^2 - 4\lambda - 12\lambda + 20 = 0$$

$$\lambda^3 + \lambda^2 - 16\lambda + 20 = 0$$

Let us again use the method discussed in Problem 6

λ must be some factor of 20

$\lambda = 1 \qquad 1 + 1 - 16 + 20 \neq 0$

$\lambda = 2 \qquad 8 + 4 - 32 + 20 = 12 + 20 - 32 = 0$

so using synthetic division

$$(\lambda - 2)(\lambda^2 + a\lambda + b) = 0$$

1	1	-16	20	$\lambda = 2$
	2	6	-20	
1	3	-10	0	

and

$$(\lambda - 2)(\lambda^2 + 3\lambda - 10) = 0$$

Solving the quadratic term appropriately gives

$$\frac{-3 \pm \sqrt{9 + 40}}{2} = \frac{-3 \pm 7}{2}$$

or

$$\lambda_2 = \frac{-3+7}{2} = 2$$

$$\lambda_3 = \frac{-3-7}{2} = -5$$

In summary

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$
$$\lambda_3 = -5$$

For $\lambda_1 = 2$ $\quad \vec{v}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} 3 & -2 & -1 \\ 3 & -4 & -3 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$3x - 2y - z = 2x$$
$$3x - 4y - 3z = 2y$$
$$2x - 4y = 2z$$

or

① $x - 2y - z = 0$

② $3x - 6y - 3z = 0$

③ $2x - 4y - 2z = 0$

Note that Eq. 2 is 3 × Eq. 1 and Eq. 3 is 2 × Eq. 1

If we set $z = 0$ in Eq. 1 we get

$$x = 2y$$

$$\text{or} \quad \vec{V}_1 = \begin{pmatrix} 2y \\ y \\ 0 \end{pmatrix} \underset{\substack{\text{For} \\ y=1}}{=} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

If we set $y = 0$ in Eq. 2 we get

$3x = 3z$ or $x = z$

$$\vec{V}_2 = \begin{pmatrix} x \\ 0 \\ x \end{pmatrix}$$

which for $x = 1$ is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

What this says is

For $\lambda_1 = 2$ $\quad \vec{V}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 2$ $\quad \vec{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

or even though λ_1 and λ_2 are equivalent eigenvalues they have two different eigenvectors!

Finally what about

$$\lambda_3 = -5$$

$$\begin{pmatrix} 3 & -2 & -1 \\ 3 & -4 & -3 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -5 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$3x - 2y - z = -5x$$
$$3x - 4y - 3z = -5y$$
$$2x - 4y = -5z$$

or

$$① \quad 8x - 2y - z = 0$$
$$② \quad 3x + y - 3z = 0$$
$$③ \quad 2x - 4y + 5z = 0$$

If we multiply Eq. ① by 5 and add the result to Eq. ③ we find

$$42x - 14y = 0$$

or

$$\boxed{3x = y}$$

Using this fact we can eliminate y from Eq. ③ to obtain

$$2x - 4y + 5z = 0$$

$$2x - 12x + 5z = 0$$

$$-10x + 5z = 0$$

$$-5z = -10x$$

$$\boxed{z = 2x}$$

Let us set $x = 1$ so

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

In summary our eigenvalues are

$$\lambda_1 = 2 \qquad \lambda_2 = 2 \qquad \lambda_3 = -5$$

and our eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Now for

$$\underset{\sim}{T} = \begin{pmatrix} 3 & -2 & -1 \\ 3 & -4 & -3 \\ 2 & -4 & 0 \end{pmatrix}$$

$$Tr\ T = 3 - 4 = -1$$

and

$$\det \underset{\sim}{T} = 3(-4)0 + (-2)(-3)(2)$$

$$(-1)(3)(-4) - (-1)(-4)2 + 2(3)0 - 9 \cdot 4 =$$

$$\det \underset{\sim}{T} = 12 + 12 - 8 - 36 = 24 - 44 = -20$$

Thus $Tr\ \underset{\sim}{T} = \lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 - 5 = -1$

$$\det \underset{\sim}{T} = \lambda_1 \lambda_2 \lambda_3 = 2(2)(-5) = -20$$

Are our eigen vectors linearly independent?

Under what circumstances is the following true?

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$$

$$c_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2c_1 + c_2 + c_3 = 0$$

$$c_1 + 0\,c_2 + 3c_3 = 0 \quad \text{or} \quad 3c_3 = 0$$

$$c_2 + 2c_3 = 0$$

Clearly $c_3 = 0$ implies $c_2 = 0$
and thus $c_1 = 0$
so our three eigenvectors
are linearly independent.!