

AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021

Zoom Lecture: Tu: 2:00-4:00 p.m.

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PROBLEM SET V (due Tuesday, June 22, 2021)

Problem 1

Use the concept of the rank of a matrix to investigate the nature of Problem 6 of Problem Set 2.

Problem 2

Use the concept of the rank of a matrix to investigate the nature of Problem 7 of Problem Set 2.

Problem 3

Use the concept of the rank of a matrix to investigate the nature of Problem 10 of Problem Set 2

Problem 4

Use the concept of the rank of a matrix to see if the following set of equations have an unique solution. If they do then use the method of **Gauss-Jordan elimination** to find this unique solution.

$$\begin{array}{rclcl} x_1 & + & x_2 & + & x_3 & = & -2 \\ x_1 & - & x_2 & + & x_3 & = & 2 \\ -x_1 & + & x_2 & - & x_3 & = & -2 \end{array}$$

Problem 5

Use the concept of the rank of a matrix to see if the following set of equations have an unique solution. If they do then use the method of **Gauss-Jordan elimination** to find this unique solution. (20 points) (Problem Set II)

$$\begin{aligned}3x - 2y &= 2 \\ -6x + 4y &= -4 \\ -3x + 2y &= 2\end{aligned}$$

Problem 6

Use the concept of the rank of a matrix to see if the following set of equations have an unique solution. If they do then use the method of **Gauss-Jordan elimination** to find this unique solution.

$$\begin{aligned}x_1 + 2x_2 - 3x_3 &= 4 \\ 2x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 2x_2 - x_3 &= 5\end{aligned}$$

Problem 7

Let

$$\vec{u} = (2, 1)$$

$$\vec{v} = (-1, 1)$$

Find the following quantities

$$(i) \quad \|\vec{u}\|$$

$$(ii) \quad \|\vec{v}\|$$

$$(iii) \quad \vec{u} + \vec{v}$$

$$(iv) \quad 2\vec{u} - 3\vec{v}$$

$$(v) \quad \|\vec{u} + \vec{v}\|$$

Problem 8

Find the unit vector in the same direction as the following vectors

$$(i) \quad (3, -1)$$

$$(ii) \quad 2\hat{i} + 3\hat{j}$$

$$(iii) \quad \hat{i} + \hat{j}$$

Problem 9

Determine the angle between the following pairs of vectors:

$$(i) \quad \vec{u} = -\hat{i} - \hat{j}$$

and

$$\vec{v} = \hat{i} + \hat{j}$$

$$(ii) \quad \vec{u} = \hat{i} + \hat{j}$$

and

$$\vec{v} = \hat{i} - \hat{j}$$

Problem 10

What angles do the following vectors make with the positive x axis?

$$(i) \quad \hat{i} + \hat{j}$$

$$(ii) \quad -\hat{i} + \hat{j}$$

$$(iii) \quad -\hat{i} - \hat{j}$$

Problem 11

Show that the two vectors

$$\vec{u} = (3, 4)$$

$$\vec{v} = (4, -3)$$

are orthogonal. Draw these two vectors in a Cartesian coordinate system.

Problem 12

Solve

$$a(1, 1) + b(-1, 0) = (1, 0)$$

for the scalars **a** and **b**.

Python Exercise 5

Let us explore the rank of a matrix with NumPy. Note that you will need some of the commands we previously learned in Python.

1. First write down an arbitrary set of six linear equations for $x_1, x_2, x_3, x_4, x_5,$ and x_6 . Do not be afraid to use real numbers instead of integers in your example.

From your equations write down a coefficient matrix **A** and a constant matrix **B**.

Here are the required commands to find the rank of a matrix in NumPy:

```
import numpy as np
A= np.array([[2, -3,1],[1,-1,2],[3,1,-1]])#here we build a test coefficient matrix
B= np.array( [[-1],[-3],[9]]) #here we build a test constant matrix where you should note
that it is 3 rows by 1 column
print (A)
print (B)
M = np.hstack([A,B])#create the augmented matrix A|B
print (M)
r = np.linalg.matrix_rank(A) # find the rank r of A
r_1 = np.linalg.matrix_rank(M) # find the rank r_1 of the augmented matrix A|B
print (r)
print (r_1)
```

In this example which I have given, is there a unique solution, an infinite number of solutions, or no solution?

Now for your example, find the rank of **A** and the rank of the augmented matrix **A|B**.

Does your linear system of equations have a unique solution, an infinite number of solutions, or no solution?

2. Repeat this exercise for a randomly generated coefficient matrix **A** (**6 x 4**) and a randomly generated constant matrix **B** (**6 x 1**)

3. Pick two of the earlier examples in this problem set and use NumPy to confirm your results. Please do this only after you have solved these problems analytically.

4. You can next use Python to simultaneously solve systems of linear equations using the **solve** function in the NumPy **linalg** subpackage. (This was just brought to my attention by an old friend, Dr. Alan D. Beckles, a retired cardiologist in New York, NY.)

```
import numpy as np
A= np.array([[2,-3,1],[1,-1,2],[3,1,-1]]) #here we build a coefficient matrix
B= np.array( [-1,-3,9]) #here we build a constant matrix and you should compare the
syntax here to what we did in Exercise 1
np.linalg.solve(A,B) #here we solve the augmented matrix to find the solution
```

Play with the previous problems in this problem set and use NumPy to confirm your results. See what NumPy tells you for the three cases when there is a unique solution, when

there are an infinite number of solutions, or when there is no solution. Please do this only after you have solved these problems analytically.

5. Use NumPy to compute the rank of matrix \mathbf{M} :

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Now use NumPy to compute the rank of matrix \mathbf{P} :

$$\mathbf{P} = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$$

Can you discover something interesting about the rank of a matrix by studying these two examples?

6. I want to leave this problem set by showing you something that will be of tremendous importance in your future studies of linear algebra. For now let us consider it a curiosity.

Construct a random 4 x 4 matrix which we will call \mathbf{U} .

Find U^T .

Find $\mathbf{G} = U^T U$.

Find $\mathbf{G} = U U^T$.

Study the matrix \mathbf{G} very carefully. What have you discovered?

Repeat these calculations starting with a wide matrix \mathbf{J} of dimension $m \times n$ where $m < n$. What have you discovered?

Repeat these calculations starting with a tall matrix \mathbf{R} of dimension $m \times n$ where $m > n$. What have you discovered?