AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021
Zoom Lecture: Tu: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET IX
(due Tuesday, July 20, 2021)

Problem 1

Describe geometrically what the following matrices do when they act on a vector:

$$(a) \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(b) \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(c) \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(d) \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(e) \quad \mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(f) \quad \mathbf{F} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(g) \qquad \mathrm{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2

Describe geometrically what the following matrices do when they act on a vector. Consider here the two possibilities that $\mathbf{k} > \mathbf{1}$ or $\mathbf{k} < \mathbf{-1}$.

$$(a) \qquad \mathrm{A} = \begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix}$$

$$(b) \qquad \mathrm{B} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$$

$$(c) \qquad \mathrm{C} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

$$(d) \qquad \mathrm{D} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$$

$$(e) \qquad \mathrm{E} = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$$

$$(f) \qquad \mathrm{F} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(g) \qquad \mathrm{G} = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$(h) \qquad \mathrm{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$(j) \quad \mathbf{J} = \begin{pmatrix} \boldsymbol{k} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{k} \end{pmatrix}$$

$$(k) \quad \mathbf{K} = \begin{pmatrix} -\boldsymbol{k} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{k} \end{pmatrix}$$

$$(l) \quad \mathbf{L} = \begin{pmatrix} \boldsymbol{k} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{k} \end{pmatrix}$$

Problem 3

Show that the following matrix **W** is orthogonal

$$\mathbf{W} = \frac{\mathbf{1}}{\mathbf{3}} \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & -\mathbf{2} \\ -\mathbf{2} & \mathbf{2} & -\mathbf{1} \end{pmatrix}$$

Problem 4

First show that **G** is orthogonal and then explicitly show that its rows (columns) are indeed orthonormal vectors.

$$\mathbf{G} = \frac{\mathbf{1}}{\mathbf{9}} \begin{pmatrix} \mathbf{1} & \mathbf{8} & -\mathbf{4} \\ \mathbf{4} & -\mathbf{4} & -\mathbf{7} \\ \mathbf{8} & \mathbf{1} & \mathbf{4} \end{pmatrix}$$

Problem 5

Show that the following matrix **T** is unitary

$$\mathbf{T} = \frac{\mathbf{1}}{\sqrt{\mathbf{3}}} \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & -\mathbf{1} & -\boldsymbol{i} \\ \mathbf{1} & -\mathbf{1} & \mathbf{0} & \boldsymbol{i} \\ \mathbf{1} & -\boldsymbol{i} & \boldsymbol{i} & \mathbf{1} \end{pmatrix}$$

Problem 6

Show that the following matrix **K** is unitary

$$\mathbf{K} = \frac{1}{6}\begin{pmatrix} 2-4i & 4i \\ -4i & -2-4i \end{pmatrix}$$

Problem 7

Show that the following matrix **Q** is unitary

$$\mathbf{Q} = \frac{1}{5}\begin{pmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{pmatrix}$$

Problem 8

Using Euler's formula, fill in the missing steps from Lecture 9 to show that the rotation matrix in two dimensions is given by

$$\mathbf{R}(\boldsymbol{\theta}) = \begin{pmatrix} \cos\boldsymbol{\theta} & -\sin\boldsymbol{\theta} \\ \sin\boldsymbol{\theta} & \cos\boldsymbol{\theta} \end{pmatrix}$$

Problem 9

As discussed in Lecture 9 prove that the rotation matrix in two dimensions obeys the following identities where **I** is the identity matrix

$$\mathbf{R}(\boldsymbol{\theta})^{\boldsymbol{T}}\mathbf{R}(\boldsymbol{\theta}) = \mathbf{R}(\boldsymbol{\theta})\mathbf{R}(\boldsymbol{\theta})^{\boldsymbol{T}} = \mathbf{I}$$

Problem 10

As discussed in Lecture 9 we shall prove here that the angle $\boldsymbol{\theta}$ between two vectors $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ is preserved when they are both acted on by an orthogonal matrix **A**.

First start with the definition of the dot product of two vectors and show that

$$\cos\boldsymbol{\theta} = \frac{\langle\vec{\boldsymbol{x}}|\vec{\boldsymbol{y}}\rangle}{\langle\vec{\boldsymbol{x}}|\vec{\boldsymbol{x}}\rangle^{\frac{1}{2}}\langle\vec{\boldsymbol{y}}|\vec{\boldsymbol{y}}\rangle^{\frac{1}{2}}}$$

Now we already know from Lecture 9 that

$$\langle\mathbf{A}\vec{\mathbf{x}}|\mathbf{A}\vec{\mathbf{x}}\rangle = \langle\vec{\mathbf{x}}|\vec{\mathbf{x}}\rangle$$

$$\langle\mathbf{A}\vec{\mathbf{y}}|\mathbf{A}\vec{\mathbf{y}}\rangle = \langle\vec{\mathbf{y}}|\vec{\mathbf{y}}\rangle$$

so we need to show that

$$\langle\mathbf{A}\vec{\mathbf{x}}|\mathbf{A}\vec{\mathbf{y}}\rangle = \langle\vec{\mathbf{x}}|\vec{\mathbf{y}}\rangle$$

Hint: Use the same approach we used in Lecture 9 to explicitly demonstrate that orthogonal matrices always preserve the length of a vector that they act on.

Python Exercise 9

1. The following is a Python script (a fancy word for a program) for plotting vectors in three dimensions

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d
fig = plt.figure()
ax = plt.axes(projection"3d")
v =[0,5,4]
ax.set_xlim([-1,10])
ax.set_ylim([-10,10])
ax.set_zlim([0,10])
start=[0,0,0]
ax.quiver(start[0], start[1], start[2], v[0],v[1],v[2])
plt.show()
```

Go back to Problem Set VI and see how you can learn new plotting skills to understand this script. All of the commands in the above script are easy to understand and you should consult on-line Python sources for more insight (e.g. try changing the colors of your vectors). Now choose a vector and use Python to visualize the vector both before and after it is acted on by the matrices in Problem 1. Slightly modify the plot script above to plot multiple vectors in a single plot. This way you can visualize both the initial and final vectors in a single plot.

2. Choose a vector and use Python to visualize the vector both before and after it is acted on by the matrices in Problem 2. Slightly modify the plot script above to plot multiple vectors in a single plot. This way you can visualize both the initial and final vectors in a single plot.