AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021
Zoom Lecture: Tu: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
Dr. Steven L. Richardson (srichards22@comcast.net)
Professor Emeritus of Electrical Engineering, Department of Electrical and Computer Engineering, Howard University, Washington, DC
and
Faculty Associate in Applied Physics, John A. Paulson School of Engineering and Applied Sciences, Harvard University, Cambridge, MA

PROBLEM SET IX
(due Tuesday, July 20, 2021)

Problem 1

Describe geometrically what the following matrices do when they act on a vector:

$$(a) \quad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(b) \quad \mathrm{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(c) \quad \mathrm{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(d) \quad \mathrm{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(e) \quad \mathrm{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(f) \quad \mathrm{F} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(g) \quad \mathrm{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2

Describe geometrically what the following matrices do when they act on a vector. Consider here the two possibilities that $\mathbf{k} > \mathbf{1}$ or $\mathbf{k} <$ **-1**.

$$(a) \quad \mathrm{A} = \begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix}$$

$$(b) \quad \mathrm{B} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$$

$$(c) \quad \mathrm{C} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

$$(d) \quad \mathrm{D} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$$

$$(e) \quad \mathrm{E} = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$$

$$(f) \quad \mathrm{F} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(g) \quad \mathrm{G} = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$(h) \quad \mathrm{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$(j) \quad \mathrm{J} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & -k \end{pmatrix}$$

$$(k) \quad \mathrm{K} = \begin{pmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$(l) \quad \mathrm{L} = \begin{pmatrix} k & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & k \end{pmatrix}$$

Problem 3

Show that the following matrix $\mathbf{W}$ is orthogonal

$$\mathbf{W} = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$$

Problem 4

First show that $\mathbf{G}$ is orthogonal and then explicitly show that its rows (columns) are indeed orthonormal vectors.

$$\mathbf{G} = \frac{1}{9} \begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix}$$

Problem 5

Show that the following matrix $\mathbf{T}$ is unitary

$$\mathrm{T} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -i \\ 1 & -1 & 0 & i \\ 1 & -i & i & 1 \end{pmatrix}$$

Problem 6

Show that the following matrix $\mathbf{K}$ is unitary

$$\mathbf{K} = \frac{1}{6}\begin{pmatrix} 2-4i & 4i \\ -4i & -2-4i \end{pmatrix}$$

Problem 7

Show that the following matrix $\mathbf{Q}$ is unitary

$$\mathbf{Q} = \frac{1}{5}\begin{pmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{pmatrix}$$

Problem 8

Using Euler's formula, fill in the missing steps from Lecture 9 to show that the rotation matrix in two dimensions is given by

$$\mathrm{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Problem 9

As discussed in Lecture 9 prove that the rotation matrix in two dimensions obeys the following identities where $\mathbf{I}$ is the identity matrix

$$\mathrm{R}(\theta)^T\mathrm{R}(\theta) = \mathrm{R}(\theta)\mathrm{R}(\theta)^T = \mathrm{I}$$

Problem 10

As discussed in Lecture 9 we shall prove here that the angle θ between two vectors $\vec{x}$ and $\vec{y}$ is preserved when they are both acted on by an orthogonal matrix $\mathbf{A}$.

First start with the definition of the dot product of two vectors and show that

$$\cos\theta = \frac{\langle\vec{x}|\vec{y}\rangle}{\langle\vec{x}|\vec{x}\rangle^{\frac{1}{2}}\langle\vec{y}|\vec{y}\rangle^{\frac{1}{2}}}$$

Now we already know from Lecture 9 that

$$\langle \mathrm{A}\vec{\mathrm{x}}|\mathrm{A}\vec{\mathrm{x}}\rangle = \langle\vec{\mathrm{x}}|\vec{\mathrm{x}}\rangle$$

$$\langle \mathrm{A}\vec{\mathrm{y}}|\mathrm{A}\vec{\mathrm{y}}\rangle = \langle\vec{\mathrm{y}}|\vec{\mathrm{y}}\rangle$$

so we need to show that

$$\langle \mathrm{A}\vec{\mathrm{x}}|\mathrm{A}\vec{\mathrm{y}}\rangle = \langle\vec{\mathrm{x}}|\vec{\mathrm{y}}\rangle$$

Hint: Use the same approach we used in Lecture 9 to explicitly demonstrate that orthogonal matrices always preserve the length of a vector that they act on.

Python Exercise 9

1. The following is a Python script (a fancy word for a program) for plotting vectors in three dimensions

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d
fig = plt.figure()
ax = plt.axes(projection"3d")
v =[0,5,4]
ax.set_xlim([-1,10])
ax.set_ylim([-10,10])
ax.set_zlim([0,10])
start=[0,0,0]
ax.quiver(start[0], start[1], start[2], v[0],v[1],v[2])
plt.show()
```

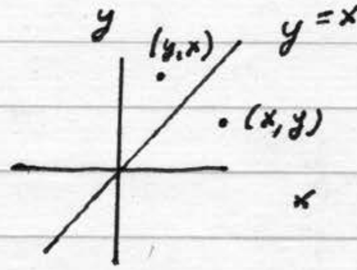
Go back to Problem Set VI and see how you can learn new plotting skills to understand this script. All of the commands in the above script are easy to understand and you should consult on-line Python sources for more insight (e.g. try changing the colors of your vectors). Now choose a vector and use Python to visualize the vector both before and after it is acted on by the matrices in Problem 1. Slightly modify the plot script above to plot multiple vectors in a single plot. This way you can visualize both the initial and final vectors in a single plot.

2. Choose a vector and use Python to visualize the vector both before and after it is acted on by the matrices in Problem 2. Slightly modify the plot script above to plot multiple vectors in a single plot. This way you can visualize both the initial and final vectors in a single plot.

Problem 1

(a) $$\underset{\sim}{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\underset{\sim}{A}\vec{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_y \\ v_x \end{pmatrix}$$



$x = v_x$

$y = v_y$

It turns $v_x \to v_y$

$v_y \to v_x$

For example

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

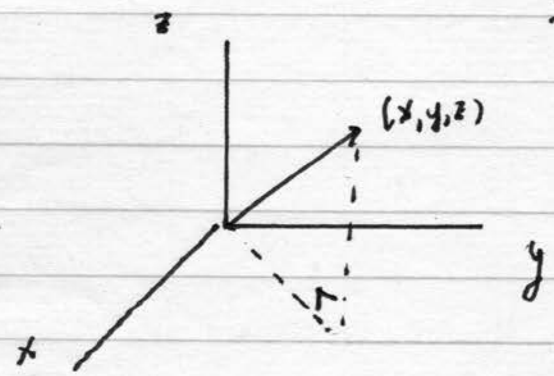
Reflection about the line $y = x$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note does not change

(b) $$\underset{\sim}{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{B}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$



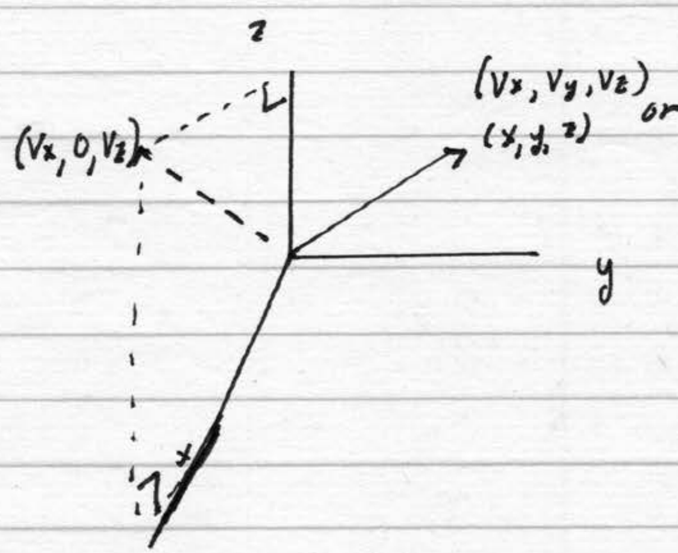
Projection on xy-plane

(c)

$$\underset{\sim}{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$\underset{\sim}{C}\vec{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x \\ 0 \\ V_z \end{pmatrix}$$

Projection on xz-plane

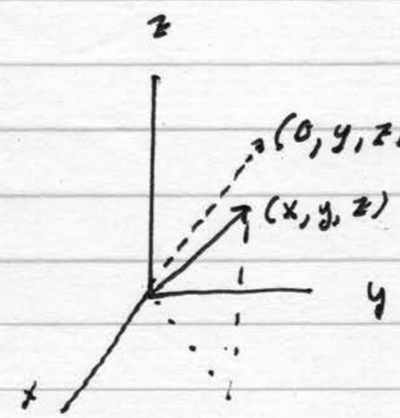


(d)

$$\underset{\sim}{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

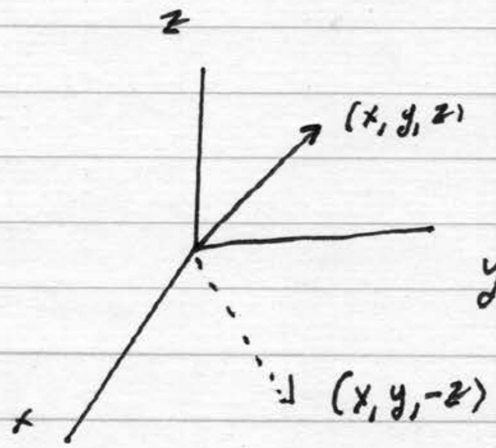
$$\underset{\sim}{D}\vec{v} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$$

Projection on yz-plane



(e) $$\underset{\sim}{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \vec{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underset{\sim}{E}\vec{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$



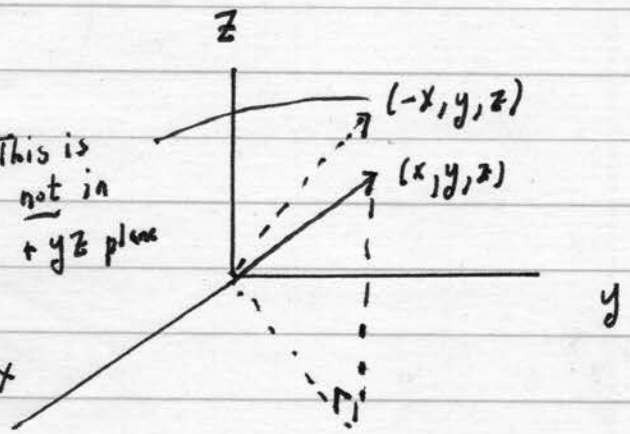
This is not in xy-plane

Reflection through xy-plane

(f) $$\underset{\sim}{F} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \qquad \vec{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underset{\sim}{F}\vec{V} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$$

This is not in +yz plane

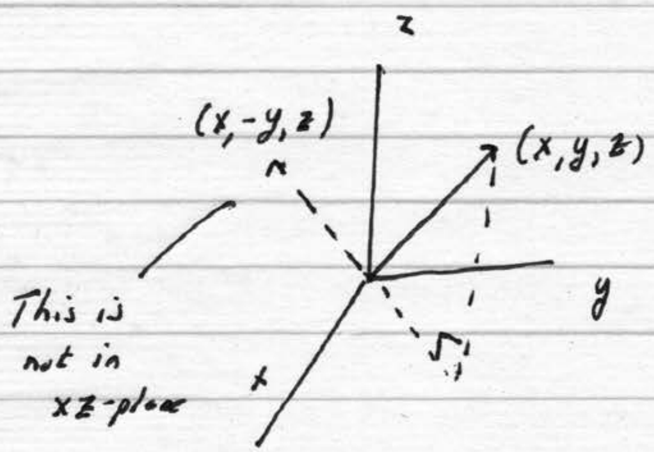


Reflection through yz-plane

(g) $$\underset{\sim}{6} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underset{\sim}{6}\,\vec{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$$

Reflection through
xz plane

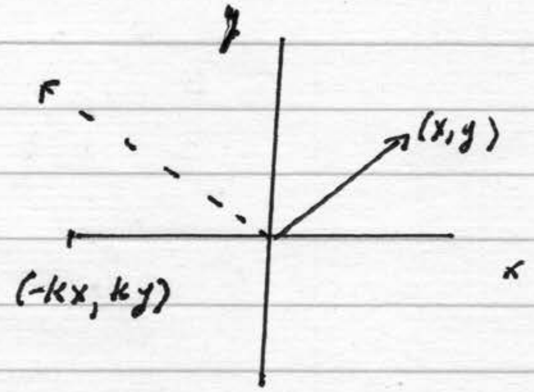


Problem 2

(a)

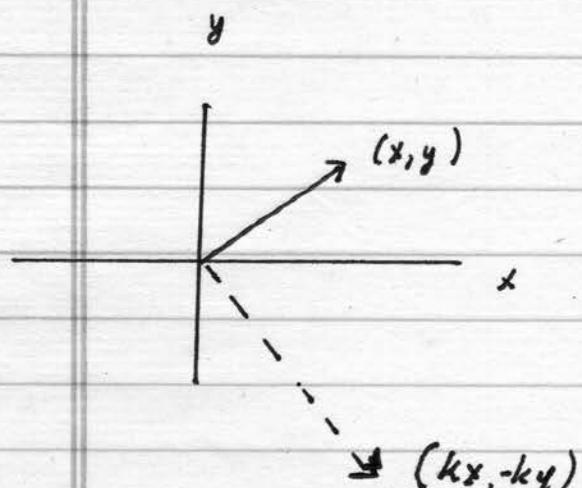
$$\underset{\sim}{A} = \begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\underset{\sim}{A}\,\vec{v} = \begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -kx \\ ky \end{pmatrix}$$

For $k > 1$, $\underset{\sim}{A}$ rotates the vector v counter-clockwise by an angle about the origin so that it lands in the second quadrant stretched by a factor of k both in $+y$ and $-x$ directions

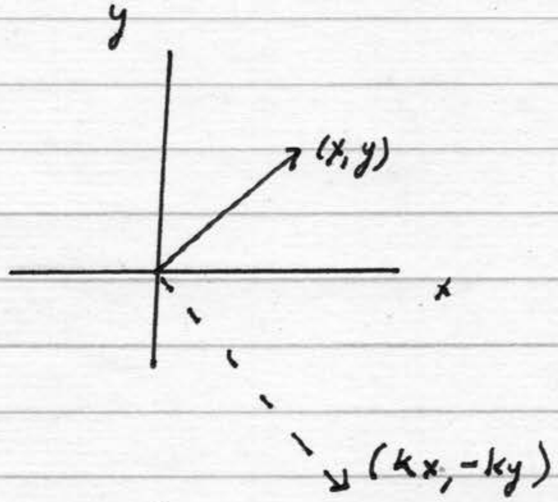


For $k < -1$, $\underset{\sim}{A}$ rotates the vector counter-clockwise by an angle about the origin so that it lands in the fourth quadrant stretched by a factor of k both in $+x$ and $-y$ directions

(1)

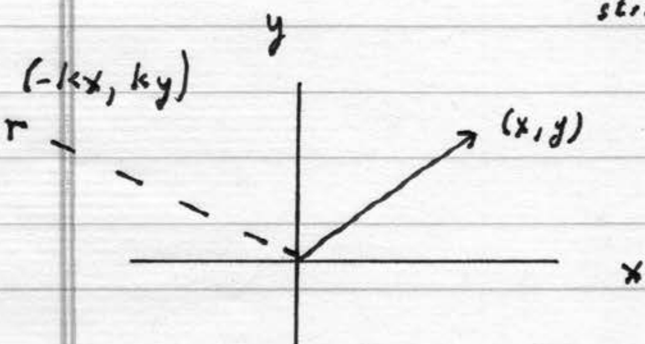
$$\underset{\sim}{B} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underset{\sim}{B}\vec{v} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ -ky \end{pmatrix}$$



N.B. This is also a reflection through +x axis followed by a stretching!

For $k > 1$, $\underset{\sim}{B}$ rotates the vector counter-clockwise by an angle about the origin so that it lands in the fourth quadrant stretched by a factor of k both in $+x$ and $-y$ directions



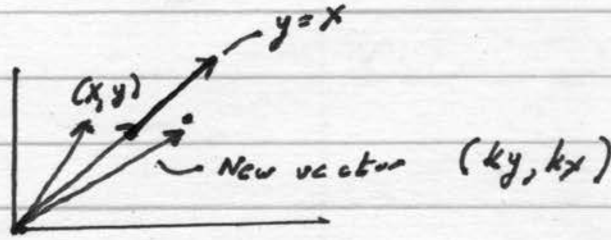
For $k < -1$, $\underset{\sim}{B}$ rotates the vector counter-clockwise by an angle about the origin so that it lands in the second quadrant stretched by a factor of k both in the $-x$ and y directions

N.B. This is also a reflection through +y axis followed by a stretching!

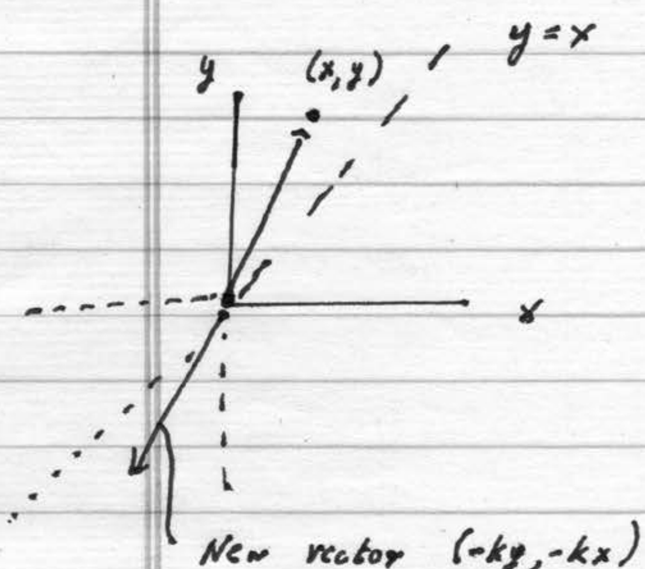
(c) $\underset{\sim}{C} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$ $\qquad \vec{V} = \begin{pmatrix} x \\ y \end{pmatrix}$

Review 1(a) to see how best to store!

$$\underset{\sim}{C}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ky \\ kx \end{pmatrix}$$



Reflects the vector $\vec{V}$ through the line $y=x$ and then stretches it so it has new coordinates (ky, kx) where $k>0$

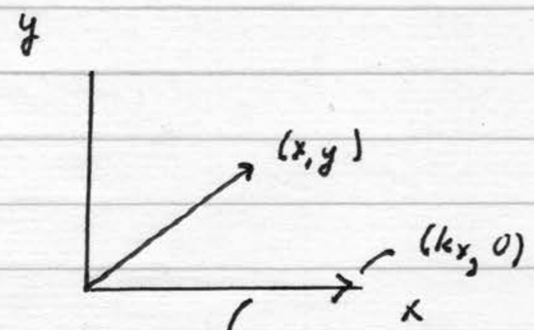


N.B. This is also an inversion through (0,0) followed by a stretching! Check out Problem 2(j)!

Rotates the vector $\vec{v}$ in a clockwise manner about origin and stretches it so it has new coordinates $(-ky, -kx)$ when $k>0$

(d) $$\underset{\sim}{D} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

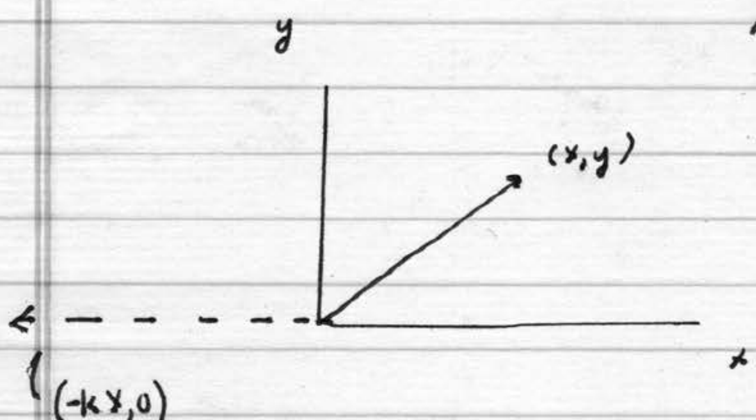
$$\underset{\sim}{D}\vec{v} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ 0 \end{pmatrix}$$



The new vector is here!

For $k > 1$, $\underset{\sim}{D}$ rotates the vector clockwise by an angle about the origin so that it lands on $+x$ axis stretched by a factor of k

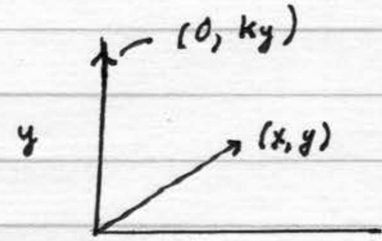
[This is also a projection and stretch!]



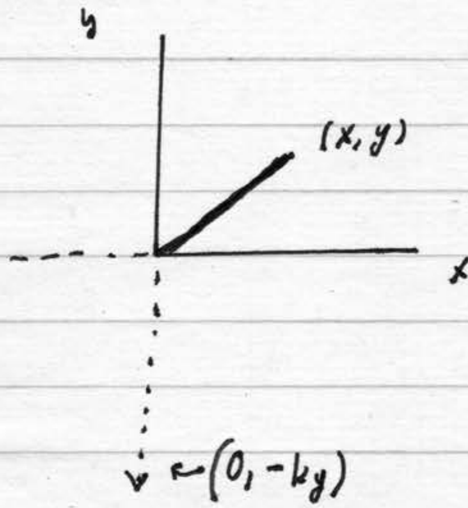
For $k < -1$, $\underset{\sim}{D}$ rotates the vector counter-clockwise by an angle about the origin so that it lands on $-x$ axis stretched by a factor of k

(e) $\underset{\sim}{E} = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$ $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\underset{\sim}{E}\vec{v} = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ ky \end{pmatrix}$$



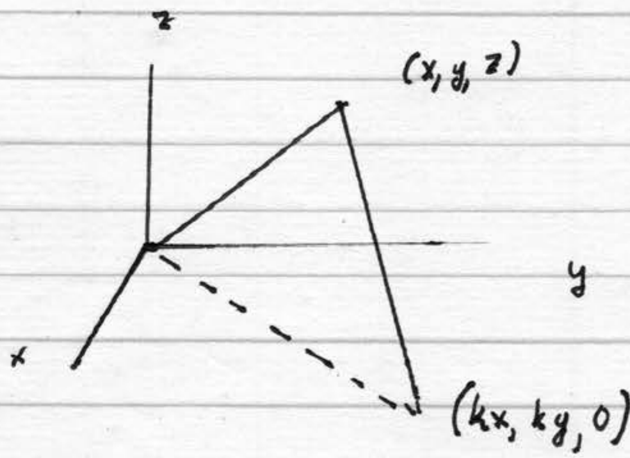
For $k > 1$ it rotates the vector counter-clockwise by a certain angle about the origin and stretches it by a factor of k along +y axis



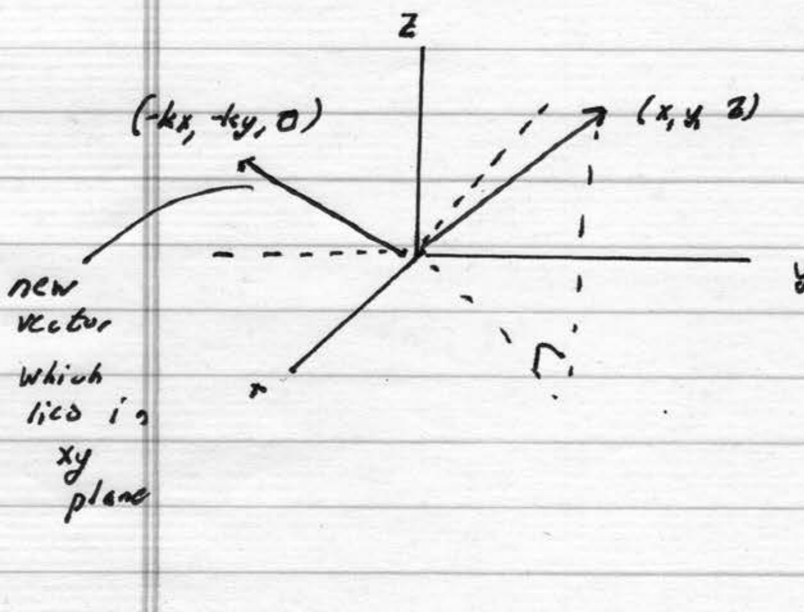
For $k < -1$ it rotates the vector counter-clockwise by a certain angle about the origin and stretches it by a factor of k along -y axis

(f) $$\underset{\sim}{F} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If we consult Problem 1(b) we can see that



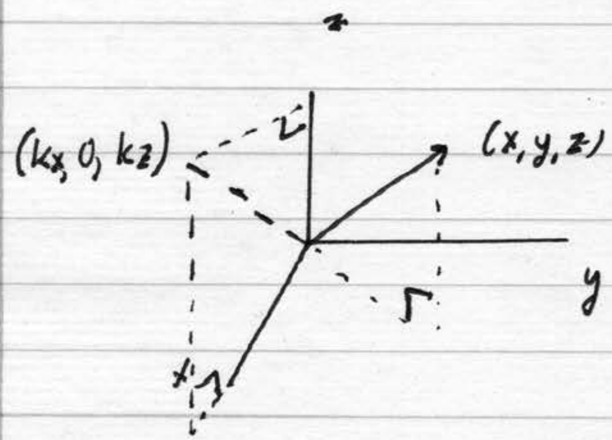
For $k>1$, $\underset{\sim}{F}$ projects $\vec{v}$ on to xy plane and stretches it by a factor of k both in +x and +y directions



For $k<-1$, $\vec{F}$ projects $\vec{v}$ on to xy plane, rotates it and then stretches it by a factor of k both in -x and -y directions

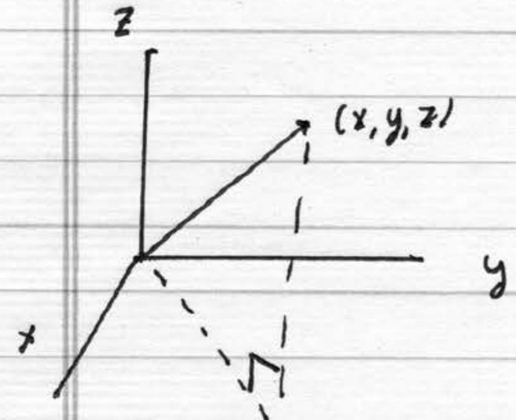
(g) $$\underset{\sim}{G} = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{pmatrix}$$

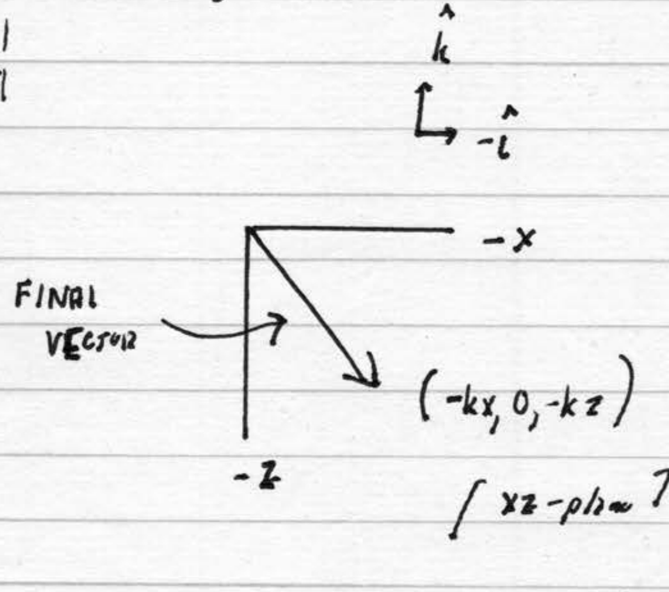
If we consult Problem 1(c) we can see that



For $k > 1$, $\underset{\sim}{G}$ projects $\vec{v}$ to xz-plane and stretches it by a factor of k both in +x and +z directions

Initial vector





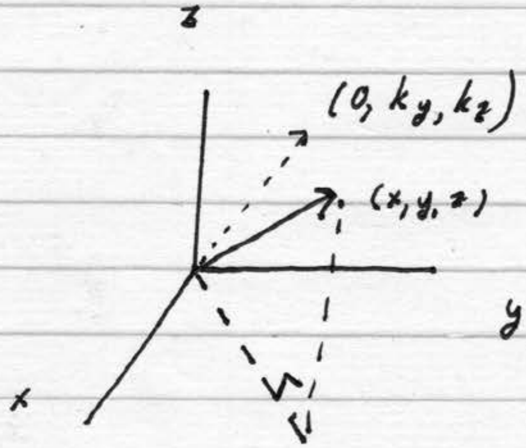
For $k < -1$, $\underset{\sim}{G}$ projects $\vec{v}$ onto the xz-plane, rotates it, and then stretches it by a factor of $k > 0$ in the -x and -z directions

(b)

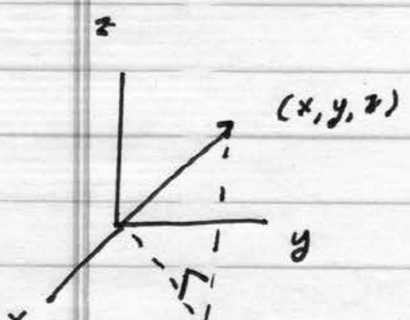
$$\underset{\sim}{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

If we consult Problem 1(d) we can see that

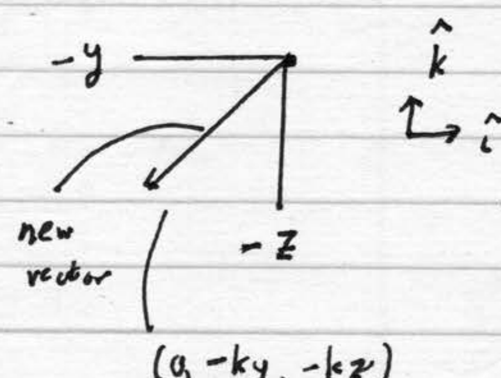
For $k>1$, $\underset{\sim}{H}$ projects $\vec{V}$ onto the yz-plane and stretches it by a factor of k both in the $+y$ and $+z$ directions



For $k<-1$, $\underset{\sim}{H}$ projects $\vec{V}$ onto the yz-plane, rotates it, and then stretches it by a factor of $k>0$ in $-y$ and $-z$ directions



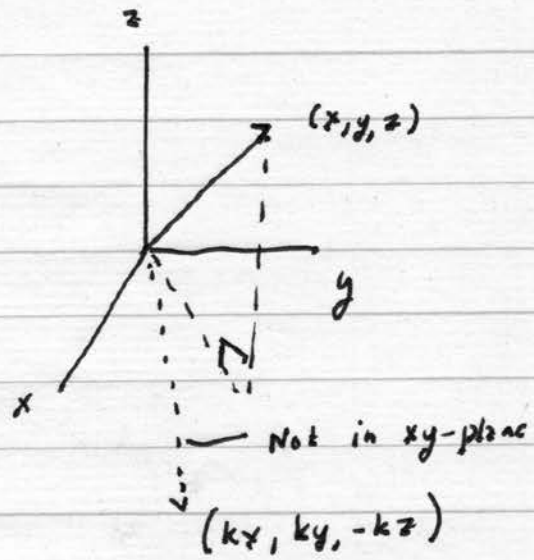
INITIAL VECTOR



FINAL VECTOR

(j) $$\underset{\sim}{J} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & -k \end{pmatrix}$$

If we consult Problem 1(c) we can see that



For $k > 1$, $\underset{\sim}{J}$ reflects $\vec{v}$ through the xy-plane and stretches it by a factor of k in $\hat{i}$ and $\hat{j}$ directions and by a factor of k in $-\hat{k}$ direction

For $k < -1$ $$\underset{\sim}{J} = \begin{pmatrix} -k & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & k \end{pmatrix}$$

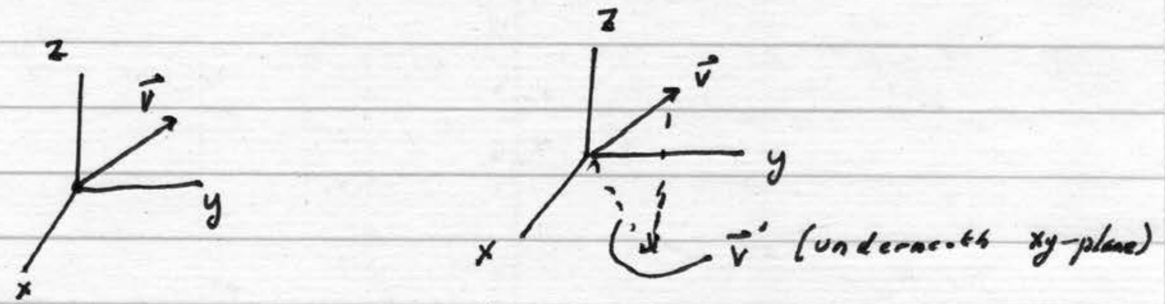
First of all the matrix $\underset{\sim}{N} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

is known as an inversion matrix since it reflects a three-dimensional vector $\vec{v}$ through the origin

$$\underset{\sim}{N} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

Recall that the matrix $\underset{\sim}{Z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

is a reflection matrix as it reflects a vector through the xy-plane



$$Z\vec{v} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

Thus

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\text{reflection through xy-plane}} \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\text{inversion through origin}}$$

Thus $\underset{\sim}{J} = \begin{pmatrix} -k & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & k \end{pmatrix}$

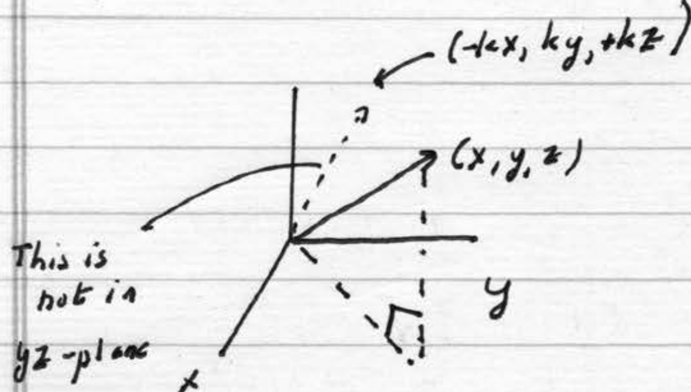
- inverts a vector through the origin and then reflects it through the xy-plane

- it finally stretches it by a factor of k in $-\hat{i}$, $-\hat{j}$ and $\hat{k}$ directions

(k)

$$\underset{\sim}{K} = \begin{pmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

If we consult 1(f) we can see that



For $k > 1$, $\underset{\sim}{K}$ reflects $\vec{v}$ through yz-plane and stretches it by a factor of k in $+\hat{j}$ and $+\hat{k}$ directions and by a factor of k in $-\hat{i}$ direction

For $k < -1$ $\underset{\sim}{K} = \begin{pmatrix} k & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & -k \end{pmatrix}$

If we consult the previous problem (2j) we easily see that

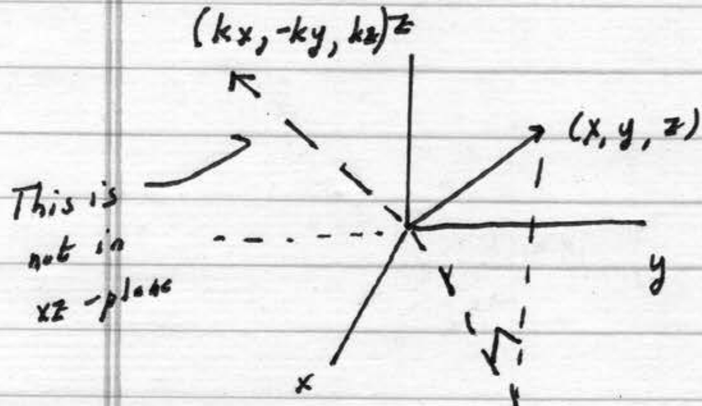
① $\underset{\sim}{K}$ inverts a vector through the origin

② Next it reflects it through yz-plane

③ Finally it stretches it by a factor of k in $-\hat{j}$ and $-\hat{k}$ directions and $+\hat{i}$ direction

(l) $\underset{\sim}{L} = \begin{pmatrix} k & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & k \end{pmatrix}$

If we consult 1(g) we can see that



For $k > 1$, $\underset{\sim}{L}$ reflects $\hat{v}$ through xz-plane and stretches it by a factor of k in $+\hat{i}$, $-\hat{j}$, and $\hat{k}$ directions

For $k < -1$ $\underset{\sim}{L} = \begin{pmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & -k \end{pmatrix}$

In a similar fashion as before let us consult problem 2(j) to see that

① $\underset{\sim}{L}$ inverts a vector through the origin

② Next it reflects it through the xz-plane

③ Finally it stretches it by a scalar k in the $+\hat{i}$, $-\hat{j}$, and $-\hat{k}$ directions

Problem 3

Prove that the following matrix $\underset{\sim}{W}$ is orthogonal

$$\underset{\sim}{W} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$$

$$\underset{\sim}{W}^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$$

$$\underset{\sim}{W}^T \underset{\sim}{W} = \frac{1}{9}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix} = \frac{1}{9}\underset{\sim}{C}$$

$$C_{11} = 1(1) + 2(2) - 2(-2) = 1 + 4 + 4 = 9$$

$$C_{12} = 1(2) + 2(1) - 2(2) = -4 + 2 + 2 = 0$$

$$C_{13} = 1(2) + 2(-2) - 2(-1) = 2 - 4 + 2 = 0$$

$$C_{21} = 2(1) + 1(2) + 2(-2) = 2 + 2 - 4 = 0$$

$$C_{22} = 2(2) + 1(1) + 2(2) = 4 + 4 + 1 = 9$$

$$C_{23} = 2(2) + 1(-2) + 2(-1) = 4 - 2 - 2 = 0$$

$$C_{31} = 2(1) + (-2)(2) + (-1)(-2) = -4 + 2 + 2 = 0$$

$$C_{32} = 2(2) + (-2)(1) + (-1)(2) = 4 - 2 - 2 = 0$$

$$C_{33} = 2(2) + (-2)(-2) + (-1)(-1) = 4 + 4 + 1 = 9$$

$$\underset{\sim}{C} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\underset{\sim}{W}^T \underset{\sim}{W} = \frac{1}{9}\underset{\sim}{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

Therefore $\underset{\sim}{W}^T = \underset{\sim}{W}^{-1}$ (at least for this equation)

$$\underset{\sim}{W}\,\underset{\sim}{W}^T = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix} = \frac{1}{9}\underset{\sim}{D}$$

$d_{11} = 1(1) + 2(2) + 2(2) = 4 + 4 + 1 = 9$

$d_{12} = 1(2) + 2(1) + 2(-2) = 2 + 2 - 4 = 0$

$d_{13} = 1(-2) + 2(2) + 2(-1) = -2 + 4 - 2 = 0$

$d_{21} = 2(1) + 1(2) - 2(2) = 2 + 2 - 4 = 0$

$d_{22} = 2(2) + 1(1) - 2(-2) = 4 + 1 + 4 = 9$

$d_{23} = 2(-2) + 1(2) - 2(-1) = -4 + 2 + 2 = 0$

$d_{31} = -2(1) + 2(2) - 1(2) = -2 + 4 - 2 = 0$

$d_{32} = -2(2) + 2(1) - 1(-2) = -4 + 2 + 2 = 0$

$d_{33} = -2(-2) + 2(2) - 1(-1) = 4 + 4 + 1 = 9$

$$\underset{\sim}{D} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\underset{\sim}{W}\,\underset{\sim}{W}^T = \frac{1}{9}\underset{\sim}{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

Therefore $\underset{\sim}{W}^T = \underset{\sim}{W}^{-1}$ (at least for this equation)

Thus

$$\underset{\sim}{W}^T\underset{\sim}{W} = \underset{\sim}{W}\,\underset{\sim}{W}^T = \underset{\sim}{I}$$

So $\underset{\sim}{W}^T = \underset{\sim}{W}^{-1}$ and $\underset{\sim}{W}$ is an orthogonal matrix!

N.B.
YOU HAVE TO PROVE BOTH OF THESE EQUATIONS WHICH WE HAVE!

Problem 4

First show that $\underset{\sim}{G}$ is orthogonal and then explicitly show that its columns (rows) are indeed orthonormal vectors.

$$\underset{\sim}{G} = \frac{1}{9}\begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix}$$

$$\underset{\sim}{G}^T = \frac{1}{9}\begin{pmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{pmatrix}$$

$$\underset{\sim}{G}^T\underset{\sim}{G} = \frac{1}{81}\begin{pmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{pmatrix}\begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix} = \frac{1}{81}\underset{\sim}{P}$$

$$P_{11} = 1(1) + 4(4) + 8(8) = 1 + 16 + 64 = 81$$
$$P_{12} = 1(8) + 4(-4) + 8(1) = 8 - 16 + 8 \quad 0$$
$$P_{13} = 1(-4) + 4(-7) + 8(4) = -4 - 28 + 32 \quad 0$$
$$P_{21} = 8(1) + -4(4) + 1(8) = 8 - 16 + 8 = 0$$
$$P_{22} = 8(8) + -4(-4) + 1(1) = 64 + 16 + 1 = 81$$
$$P_{23} = 8(-4) + -4(-7) + 1(4) = -32 + 28 + 4 = 0$$
$$P_{31} = -4(1) \quad -7(4) + 4(8) = 32 - 4 - 28 = 0$$
$$P_{32} = -4(8) \quad -7(-4) + 4(1) = -32 + 4 + 28 = 0$$
$$P_{33} = -4(-4) \quad -7(-7) + 4(4) = 16 + 16 + 49 = 81$$

$$\underset{\sim}{P} = \begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix}$$

$$\underset{\sim}{G}^T \underset{\sim}{G} = \frac{1}{81} \underset{\sim}{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

$$\underset{\sim}{G} \, \underset{\sim}{G}^T = \frac{1}{81} \begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{pmatrix} = \frac{1}{81} \underset{\sim}{S}$$

$$S_{11} = 1(1) + 8(8) - 4(-4) = 1 + 64 + 16 = 81$$

$$S_{12} = 1(4) + 8(-4) - 4(-7) = +28 - 32 + 4 = 0$$

$$S_{13} = 1(8) + 8(1) + -4(4) = 8 + 8 - 16 = 0$$

$$S_{21} = 4(1) + -4(8) - 7(-4) = 4 + 28 - 32 = 0$$

$$S_{22} = 4(4) - 4(-4) - 7(-7) = 16 + 16 + 49 = 81$$

$$S_{23} = 4(8) - 4(1) - 7(4) = 32 - 4 - 28 = 0$$

$$S_{31} = 8(1) + 1(8) + 4(-4) = 8 + 8 - 16 = 0$$

$$S_{32} = 8(4) + 1(-4) + 4(-7) = 32 - 4 - 28 = 0$$

$$S_{33} = 8(8) + 1(1) + 4(4) = 8 \cdot 8 + 16 + 1 = 81$$

$$\underset{\sim}{S} = \begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix}$$

$$\underset{\sim}{G}\,\underset{\sim}{G}^T = \frac{1}{81}\underset{\sim}{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

$$\underset{\sim}{G}\,\underset{\sim}{G}^T = \underset{\sim}{I}$$

Therefore $\underset{\sim}{G}\,\underset{\sim}{G}^T = \underset{\sim}{G}^T\underset{\sim}{G} = \underset{\sim}{I}$

and $\underset{\sim}{G}^T = \underset{\sim}{G}^{-1}$ so $\underset{\sim}{G}$ is orthogonal

$$\underset{\sim}{C} = \frac{1}{9}\begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix}$$

$\vec{c}_1 = (1, 4, 8)$

$\vec{c}_2 = (8, -4, 1)$

$\vec{c}_1 \cdot \vec{c}_2 = 8 - 16 + 8 = 0 = \vec{c}_2 \cdot \vec{c}_1$ ✓ ✓

$\vec{c}_3 = (-4, -7, 4)$

$\vec{c}_1 \cdot \vec{c}_3 = -4 - 28 + 32 = 0 = \vec{c}_3 \cdot \vec{c}_1$ ✓ ✓

$\vec{c}_2 \cdot \vec{c}_3 = 8(-4) + (-4)(-7) + (1)(4)$ ✓

$= -32 + 28 + 4 = 0 = \vec{c}_3 \cdot \vec{c}_2$ ✓

$\vec{r}_1 = (1, 8, -4)$

$\vec{r}_2 = (4, -4, -7)$

$\vec{r}_1 \cdot \vec{r}_2 = 4 - 32 + 28 = 0 = \vec{r}_2 \cdot \vec{r}_1$ ✓ ✓

$\vec{r}_3 = (8, 1, 4)$

$\vec{r}_1 \cdot \vec{r}_3 = (8) + 8 - 16 = 0 = \vec{r}_3 \cdot \vec{r}_1$ ✓ ✓

$\vec{r}_2 \cdot \vec{r}_3 = 32 - 4 - 28 = 0 = \vec{r}_3 \cdot \vec{r}_2$ ✓ ✓

Thus all rows and columns
of B are orthogonal

Note that in our previous calculations we ignored the factor of 1/9 but our results are correct independent of this constant!

Now

$$\vec{c}_1 = \frac{1}{9}(1, 4, 8)$$

$$\vec{c}_1 \cdot \vec{c}_1 = \frac{1}{81}(1 + 16 + 64) = 1$$

$$\|\vec{c}_1\| = (\vec{c}_1 \cdot \vec{c}_1)^{1/2} = 1$$

$$\vec{c}_2 = \frac{1}{9}(8, -4, +1)$$

$$\vec{c}_2 \cdot \vec{c}_2 = \frac{1}{81}(64 + 16 + 1) = 1$$

$$\|\vec{c}_2\| = (\vec{c}_2 \cdot \vec{c}_2)^{1/2} = 1$$

$$\vec{c}_3 = \frac{1}{9}(-4, -7, 4)$$

$$\vec{c}_3 \cdot \vec{c}_3 = \frac{1}{81}(16 + 49 + 16) = 1$$

$$\|\vec{c}_3\| = 1$$

$$\vec{r}_2 = \frac{1}{9}\left(4, -4, -7\right)$$

$$\vec{r}_2 \cdot \vec{r}_2 = \frac{1}{81}\left(16 + 16 + 49\right) = 1$$

$$\|\vec{r}_2\| = \left(\vec{r}_2 \cdot \vec{r}_2\right)^{1/2} = 1$$

$$\vec{r}_1 = \frac{1}{9}\left(1, 8, -4\right)$$

$$\vec{r}_1 \cdot \vec{r}_1 = \frac{1}{81}\left(1 + 64 + 16\right) = 1$$

$$\|\vec{r}_1\| = \left(\vec{r}_1 \cdot \vec{r}_1\right)^{1/2} = 1$$

$$\vec{r}_3 = \frac{1}{9}\left(8, 1, 4\right)$$

$$\vec{r}_3 \cdot \vec{r}_3 = \frac{1}{81}\left(64 + 16 + 1\right) = 1$$

$$\|\vec{r}_3\| = \left(\vec{r}_3 \cdot \vec{r}_3\right)^{1/2} = 1$$

Thus all rows and columns of $\underset{\sim}{G}$ have norms of 1.

In summary all rows and columns of $\vec{G}$ are orthonormal!

Problem 5

Show that $\underset{\sim}{T}$ is a unitary matrix

$$\underset{\sim}{T} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -i \\ 1 & -1 & 0 & i \\ 0 & -i & i & 1 \end{pmatrix}$$

$$\underset{\sim}{T}^{\dagger} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & i \\ 1 & -1 & 0 & -i \\ 0 & i & -i & 1 \end{pmatrix}$$

$$\underset{\sim}{T}\,\underset{\sim}{T}^{\dagger} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -i \\ 1 & -1 & 0 & i \\ 0 & -i & i & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & i \\ 1 & -1 & 0 & -i \\ 0 & i & -i & 1 \end{pmatrix} = \frac{1}{3} \underset{\sim}{B}$$

$$b_{11} = 1(1) + 1(1) + 1(1) + 0(0) = 3$$

$$b_{12} = 1(1) + 1(0) + 1(-1) + 0(i) = 1 - 1 = 0$$

$$b_{13} = 1(1) + 1(-1) + 1(0) + 0(-i) = 0$$

$$b_{14} = 1(0) + 1(i) + 1(-i) + 0(1) = i - i = 0$$

$$b_{21} = 1(1) + 0(1) - 1(1) - i(0) = 1 - 1 = 0$$

$$b_{22} = 1(1) + 0(0) - 1(-1) - i(i) = 1 + 1 + 1 = 3$$

$$b_{23} = 1(1) + 0(-1) - 1(0) - i(-i) = 1 - 1 = 0$$

$$b_{24} = 1(0) + 0(i) - 1(-i) - i(1) = -i + i = 0$$

$b_{31} = 1(1) + -1(1) + 0(1) + i(0) = 1-1 = 0$

$b_{32} = 1(1) \quad -1(0) + 0(-1) + i(i) \quad 1-1 = 0$

$b_{33} = 1(1) \quad -1(-1) + 0(0) \quad i(-i) = 1+1+1 = 3$

$b_{34} = 1(0) + -1(i) + 0(-i) + i(1) = -i + i = 0$

$b_{41} = 0(1) \quad -i(1) + i(1) + \quad 1(0) = i - i = 0$

$b_{42} = 0(1) \quad -i(0) + i(-1) + 1(i) = i - i = 0$

$b_{43} = 0(1) \quad -i(-1) + i(0) + 1(-i) = -i + i = 0$

$b_{44} = 0(0) + -i(i) + i(-i) + 1(1) = 1+1+1 = 3$

$$\underset{\sim}{B} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\underset{\sim}{T}\,\underset{\sim}{T}^{\dagger} = \tfrac{1}{3}\,\underset{\sim}{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

$$\underset{\sim}{T}\,\underset{\sim}{T}^{\dagger} = \underset{\sim}{I}$$

$$\underset{\sim}{T}^{\dagger}\underset{\sim}{T} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & i \\ 1 & -1 & 0 & -i \\ 0 & i & -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -i \\ 1 & -1 & 0 & i \\ 0 & -i & i & 1 \end{pmatrix} = \tfrac{1}{3}\underset{\sim}{P}$$

$$p_{11} = 1(1) + 1(1) + 1(1) + 0(0) = 3$$
$$p_{12} = 1(1) + 1(0) + 1(-1) + 0(-i) = 1-1 = 0$$
$$p_{13} \quad 1(1) + 1(-1) + 1(0) + 0(i) = 1-1 = 0$$
$$p_{14} = 1(0) + 1(-i) + 1(i) + 0(1) = i-i = 0$$

$$p_{21} = 1(1) + 0(1) \quad -1(1) + i(0) = 1-1 = 0$$
$$p_{22} = 1(1) + 0(0) + -1(-1) + i(-i) \quad 1+1+1 = 3$$
$$p_{23} = 1(1) + 0(-1) \quad -1(0) + i(i) = -1+1 = 0$$
$$p_{24} = 1(0) + 0(-i) \quad -1(i) + i(1) = i-i = 0$$

$$p_{31} = 1(1) \quad -1(1) + 0(1) \quad -i(0) = 1-1 = 0$$
$$p_{32} = 1(1) + -1(0) + 0(-1) \quad -i(-i) = -1+1 = 0$$
$$p_{33} = 1(1) \quad -1(-1) + 0(0) \quad -i(i) = 1+1+1 = 3$$
$$p_{34} = 1(0) + -1(-i) + 0(i) \quad -i(1) = -i+i = 0$$

$$p_{41} = 0(1) + i(1) \quad -i(1) + 1(0) = i-i = 0$$
$$p_{42} = 0(1) + i(0) \quad -i(-1) + 1(-i) = -i+i = 0$$
$$p_{43} = 0(1) + i(-1) \quad -i(0) + 1(i) = i-i = 0$$
$$p_{44} = 0(0) + i(-i) \quad -i(i) + 1(1) = 1+1+1 = 3$$

$$\underset{\sim}{P} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\underset{\sim}{T}^{+}\underset{\sim}{T} = \tfrac{1}{3}\underset{\sim}{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

so

$$\underset{\sim}{T}^{+}\underset{\sim}{T} = \underset{\sim}{T}\,\underset{\sim}{T}^{+} = \underset{\sim}{I}$$

and $\underset{\sim}{T}^{+} = \underset{\sim}{T}^{-1}$ so $\underset{\sim}{T}$ is a <u>unitary matrix</u>

Problem 6

Show that the following matrix $\underset{\sim}{K}$ is unitary

$$\underset{\sim}{K} = \frac{1}{6}\begin{pmatrix} 2-4i & 4i \\ -4i & -2-4i \end{pmatrix}$$

$$\underset{\sim}{K}^T = \frac{1}{6}\begin{pmatrix} 2-4i & -4i \\ 4i & -2-4i \end{pmatrix}$$

$$\left(\underset{\sim}{K}^T\right)^* = \frac{1}{6}\begin{pmatrix} 2+4i & +4i \\ -4i & -2+4i \end{pmatrix} = \underset{\sim}{K}^\dagger$$

$$\underset{\sim}{K}^* = \frac{1}{6}\begin{pmatrix} 2+4i & -4i \\ 4i & -2+4i \end{pmatrix}$$

$$\left(\underset{\sim}{K}^*\right)^T = \frac{1}{6}\begin{pmatrix} 2+4i & 4i \\ -4i & -2+4i \end{pmatrix} = \underset{\sim}{K}^\dagger$$

Note these are identical

Now we need to show $\underset{\sim}{K}^\dagger \underset{\sim}{K} = \underset{\sim}{K}\,\underset{\sim}{K}^\dagger = \underset{\sim}{I}$

$$\underset{\sim}{K}^{\dagger}\underset{\sim}{K} = \frac{1}{36}\begin{pmatrix} 2+4i & 4i \\ -4i & -2+4i \end{pmatrix}\begin{pmatrix} 2-4i & 4i \\ -4i & -2-4i \end{pmatrix}$$

$$\underset{\sim}{K}^{\dagger}\underset{\sim}{K} = \frac{1}{36}\begin{pmatrix} (2+4i)(2-4i) + 4i(-4i) & 4i(2+4i) + 4i(-2-4i) \\ -4i(2-4i) + (-i4)(-2+4i) & 4i(-4i) + (-2-4i)(-2+4i) \end{pmatrix}$$

$(-2+4i)$

$$\underset{\sim}{K}^{\dagger}\underset{\sim}{K} = \frac{1}{36}\begin{pmatrix} 4+16+16 & 8i-16 \quad -8i+16 \\ -16-8i \quad +8i+16 & 16 \quad +4+8i-8i+16 \end{pmatrix}$$

$$\underset{\sim}{K}^{\dagger}\underset{\sim}{K} = \frac{1}{36}\begin{pmatrix} 36 & 0 \\ 0 & 36 \end{pmatrix} = \underset{\sim}{I}$$

Now

$$\underset{\sim}{K}\,\underset{\sim}{K}^{\dagger} = \frac{1}{36}\begin{pmatrix} 2-4i & 4i \\ -4i & -2-4i \end{pmatrix}\begin{pmatrix} 2+4i & 4i \\ -4i & -2+4i \end{pmatrix}$$

$$\underset{\sim}{K}\,\underset{\sim}{K}^{\dagger} = \frac{1}{36}\begin{pmatrix} (2+4i)(2-4i) + 4i(-4i) & 4i(2-4i) + 4i(4i-2) \\ -4i(2+4i) - 4i(-2-4i) & 4i(-4i) + (4i-2)(-2-4i) \end{pmatrix}$$

$$\underset{\sim}{K}\,\underset{\sim}{K}^{\dagger} = \frac{1}{36}\begin{pmatrix} 4+16+16 & 8i+16+16(-1)-8i \\ -8i+16+8i+(-16) & 16 + -8i + 8i + 4 + 16 \end{pmatrix}$$

$$\underset{\sim}{K}\,\underset{\sim}{K}^{\dagger} = \frac{1}{36}\begin{pmatrix} 36 & 0 \\ 0 & 36 \end{pmatrix} = \underset{\sim}{I}$$

Thus $\underset{\sim}{K}^{\dagger}\underset{\sim}{K} = \underset{\sim}{K}\,\underset{\sim}{K}^{\dagger} = \underset{\sim}{I}$

and

$\underset{\sim}{K}^{\dagger} = \underset{\sim}{K}^{-1}$

so $\underset{\sim}{K}$ is unitary

Q.E.D.

<u>Problem 7</u>

<u>Show that the following matrix $\underset{\sim}{Q}$ is unitary</u>

$$Q = \frac{1}{5} \begin{pmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{pmatrix}$$

$$Q^{+} = \frac{1}{5} \begin{pmatrix} -1-2i & 2+4i \\ -4+2i & -2+i \end{pmatrix}$$

$$\underset{\sim}{Q}^{+} \underset{\sim}{Q} = \frac{1}{25} \begin{pmatrix} -1-2i & 2+4i \\ -4+2i & -2+i \end{pmatrix} \begin{pmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{pmatrix}$$

$$= \frac{1}{25} \underset{\sim}{K}$$

$$k_{11} = (-1-2i)(-1+2i) + (2+4i)(2-4i) =$$

$$1+2i-2i+4+4+16 = 25$$

$$k_{12} = (-1-2i)(-4-2i) + (2+4i)(-2-i) =$$

$$4-4+8i+2i-4-8i+4$$

$$-2i$$

$$= 0$$

$$k_{21} = (-4+2i)(-1+2i) + (-2+i)(2-4i)$$

$$= \cancel{4} \cancel{-2i} - 8i \cancel{-4} \quad \cancel{-4} \cancel{+4} \cancel{+2i} + 8i = 0$$

$$k_{22} = (-4+2i)(-4-2i) + (-2+i)(-2-i) =$$

$$\underline{16} - 8i + 8i + \underline{4} \quad \underline{+4} - 2i + 2i \quad \underline{+1} = 25$$

$$\underset{\sim}{K} = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

$$\underset{\sim}{Q}^{+}\underset{\sim}{Q} = \frac{1}{25}\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

$$\underset{\sim}{Q}\,\underset{\sim}{Q}^{+} = \frac{1}{25}\begin{pmatrix} -1+2i & -4-2i \\ 2-4i & -2-i \end{pmatrix}\begin{pmatrix} -1-2i & 2+4i \\ -4+2i & -2+i \end{pmatrix}$$

$$= \frac{1}{25}\underset{\sim}{N}$$

$$n_{11} = (-1+2i)(-1-2i) + (-4-2i)(-4+2i)$$

$$= 1 - 2i + 2i + 4 + 16 + 4 + 8i - 8i = 25$$

$$n_{12} = (-1+2i)(2+4i) + (-4-2i)(-2+i)$$

$$= -2 + 4i - 4i - 8 + 8 + 4i - 4i + 2 = 0$$

$$n_{21} = (2-4i)(-1-2i) + (-2-i)(-4+2i)$$

$$= -2 + 4i + 8(-1) - 2i + 8 + 2 + 4i - 4i = 0$$

$$n_{22} = (2-4i)(2+4i) + (-2-i)(-2+i)$$

$$= 4 + 16 - 8i + 8i + 4 + 1 + 2i - 2i = 25$$

$$\underset{\sim}{N} = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$$

$$\underset{\sim}{Q}\, Q^{\dagger} = \frac{1}{25} N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

Since $\underset{\sim}{Q}^{\dagger} \underset{\sim}{Q} = \underset{\sim}{Q}\,\underset{\sim}{Q}^{\dagger} = \underset{\sim}{I} \Rightarrow \underset{\sim}{Q}^{\dagger} = \underset{\sim}{Q}^{-1}$ and $\underset{\sim}{Q}$ is unitary

Problem 8

Using Euler's formula, fill in the missing steps from Lecture 9 to show that the rotation matrix in two dimensions is given by

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

From Lecture 9

$$x_1 = r\cos\alpha$$
$$x_2 = r\sin\alpha$$

$$x_2 = r\cos(\alpha+\theta)$$
$$y_2 = r\sin(\alpha+\theta)$$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

so $e^{i\alpha} = \cos\alpha + i\sin\alpha$

$$e^{i\alpha}e^{i\theta} = e^{i(\alpha+\theta)} = (\cos\theta + i\sin\theta)(\cos\alpha + i\sin\alpha)$$

$$e^{i(\alpha+\theta)} = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$
$$+ i\sin\theta\cos\alpha + i\sin\alpha\cos\theta$$

$$e^{i(\alpha+\theta)} = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$+ i\left(\sin\theta\cos\alpha + \sin\alpha\cos\theta\right)$$

Thus

$$e^{i(\alpha+\theta)} = \cos(\alpha+\theta) + i\sin(\alpha+\theta) =$$

$$\cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$+ i(\sin\theta\cos\alpha + \sin\alpha\cos\theta)$$

and if we equate the real and imaginary parts

$$\cos(\alpha+\theta) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$\sin(\alpha+\theta) = \sin\theta\cos\alpha + \sin\alpha\cos\theta$$

Therefore

$$x_1 = r\cos\alpha$$

$$x_2 = r\sin\alpha$$

$$X_2 = r\cos(\alpha+\theta) = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$$

$$x_2 = x_1\cos\theta - x_2\sin\theta$$

$$y_2 = r\sin(\alpha+\theta) = r\sin\theta\cos\alpha$$

$$+ r\sin\alpha\cos\theta$$

$$y_2 = x_1\sin\theta + x_2\cos\theta$$

Pulling all of this together gives

$$x_2 = x_1 \cos\theta - x_2 \sin\theta$$

$$y_2 = x_1 \sin\theta + x_2 \cos\theta$$

which we can express in the form of a matrix equation

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

which is our desired result

$$\underset{\sim}{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Problem 9

As discussed in Lecture 9 prove that the rotation matrix in two dimensions obeys the following identities where $\underset{\sim}{I}$ is the identity matrix

$$\underset{\sim}{R}^T(\theta)\,\underset{\sim}{R}(\theta) = \underset{\sim}{R}(\theta)\,\underset{\sim}{R}(\theta)^T = \underset{\sim}{I}$$

Given $\underset{\sim}{R}(\theta)$ from the previous problem

$$\underset{\sim}{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\underset{\sim}{R}^T(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\underset{\sim}{R}(\theta)\,\underset{\sim}{R}^T(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

$$\underset{\sim}{R}^T(\theta)\,\underset{\sim}{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

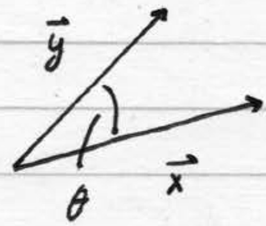
$$\underset{\sim}{R}^T(\theta)\,\underset{\sim}{R}(\theta) = \begin{pmatrix} \sin^2\theta + \cos^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$\underset{\sim}{R}^T(\theta)\,\underset{\sim}{R}(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{\sim}{I}$$

Thus $\underset{\sim}{R}^T(\theta)\,\underset{\sim}{R}(\theta) = \underset{\sim}{R}(\theta)\,\underset{\sim}{R}^T(\theta) = \underset{\sim}{I}$

Q.E.D.

Problem 10



Let us start with

$$\vec{x}\cdot\vec{y} = \|\vec{x}\|\,\|\vec{y}\|\cos\theta \qquad \langle\vec{x}|\vec{x}\rangle^{1/2}\,\langle\vec{y}|\vec{y}\rangle^{1/2}\cos\theta$$

so

$$\cos\theta = \frac{\langle\vec{x}|\vec{y}\rangle}{\langle\vec{x}|\vec{x}\rangle^{1/2}\,\langle\vec{y}|\vec{y}\rangle^{1/2}} \qquad (10\text{-}1)$$

We need to show that

$$(10\text{-}2) \qquad \frac{\langle \underset{\sim}{A}\vec{x}\,|\,\underset{\sim}{A}\vec{y}\rangle}{\langle \underset{\sim}{A}\vec{x}\,|\,\underset{\sim}{A}\vec{x}\rangle^{1/2}\,\langle \underset{\sim}{A}\vec{y}\,|\,\underset{\sim}{A}\vec{y}\rangle^{1/2}} = \cos\theta$$

We already know from Lecture 9 that

$$(10\text{-}3) \qquad \langle \underset{\sim}{A}\vec{x}\,|\,\underset{\sim}{A}\vec{x}\rangle = \langle\vec{x}|\vec{x}\rangle$$

$$(10\text{-}4) \qquad \langle \underset{\sim}{A}\vec{y}\,|\,\underset{\sim}{A}\vec{y}\rangle = \langle\vec{y}|\vec{y}\rangle$$

so (10-5) becomes

$$(10\text{-}6) \qquad \frac{\langle \underset{\sim}{A}\vec{x}\,|\,\underset{\sim}{A}\vec{y}\rangle}{\langle\vec{x}|\vec{x}\rangle^{1/2}\,\langle\vec{y}|\vec{y}\rangle^{1/2}}$$

which is almost $\cos\theta$

We need to show

$$(10-7) \qquad \langle \underset{\sim}{A}\vec{x} \,|\, \underset{\sim}{A}\vec{y} \rangle = \langle \vec{x} \,|\, \vec{y} \rangle$$

so let us proceed

$$\langle \underset{\sim}{A}\vec{x} \,|\, \underset{\sim}{A}\vec{y} \rangle = (\underset{\sim}{A}\vec{x}) \cdot (\underset{\sim}{A}\vec{y}) = (\underset{\sim}{A}\vec{x})^T (\underset{\sim}{A}\vec{y})$$

$$\langle \underset{\sim}{A}\vec{x} \,|\, \underset{\sim}{A}\vec{y} \rangle = \vec{x}^T \underset{\sim}{A}^T \underset{\sim}{A} \vec{y} = \vec{x}^T \vec{y} \quad \left(\text{provided } \underset{\sim}{A}^T \underset{\sim}{A} = \underset{\sim}{I}\right)$$

$$\langle \underset{\sim}{A}\vec{x} \,|\, \underset{\sim}{A}\vec{y} \rangle = \vec{x} \cdot \vec{y} = \langle \vec{x} \,|\, \vec{y} \rangle$$

Q.E.D.

Note that $\underset{\sim}{A}^T \underset{\sim}{A} = \underset{\sim}{I} \Rightarrow (\underset{\sim}{A}^T \underset{\sim}{A})^T = \underset{\sim}{I}^T$

$$= \underset{\sim}{A}\,\underset{\sim}{A}^T = \underset{\sim}{I}$$

which means $\underset{\sim}{A}^T = \underset{\sim}{A}^{-1}$ (10-8)

or $\underset{\sim}{A}$ is an orthogonal matrix.

Q.E.D.