AN INTRODUCTION TO LINEAR ALGEBRA USING PYTHON

Summer 2021
Zoom Lecture: Tu: 2:00-4:00 p.m.
National Science Foundation (NSF) Center for Integrated Quantum Materials (CIQM), DMR -1231319
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PROBLEM SET III
(due Tuesday, June 8, 2021)

Problem 1

Use the method of **Gauss-Jordan elimination** to solve the following equations:

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & 3 \\ 2x_1 & - & 4x_2 & = & 6 \\ -3x_1 & + & 6x_2 & = & -9 \end{array}$$

Problem 2

Use the method of **Gauss-Jordan elimination** to solve the following equations:

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 5 \\ 2x_1 & + & 3x_2 & - & x_3 & - & 2x_4 & = & 2 \\ 4x_1 & + & 5x_2 & + & 3x_3 & & & = & 7 \end{array}$$

Problem 3

Given the two matrices

$$\mathrm{A} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\mathrm{B} = \begin{pmatrix} -1 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

form the matrices $\mathbf{C} = \mathbf{2A} - \mathbf{3B}$ the matrices $\mathbf{D} = \mathbf{6B} - \mathbf{A}$.

Problem 4

Show that for the matrices defined in Problem 3: (i) $(\mathbf{A}^T)^T = \mathbf{A}$; and (ii) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Problem 5

Given the three matrices

$$\mathrm{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathrm{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\mathrm{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

show that : $\mathrm{AB} - \mathrm{BA} = \mathrm{iC}$, $\mathrm{BC} - \mathrm{CB} = \mathrm{iA}$, $\mathrm{CA} - \mathrm{AC} = \mathrm{iB}$, $\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 = 2\,\mathrm{I}$.

Problem 6

Given the three matrices

$$\mathrm{A} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{B} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathrm{C} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

show that : $\mathrm{AB} - \mathrm{BA} = \mathrm{iC}$, $\mathrm{BC} - \mathrm{CB} = \mathrm{iA}$, $\mathrm{CA} - \mathrm{AC} = \mathrm{iB}$, $\mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 = \frac{3}{4}\mathrm{I}$.

Problem 7

Given the matrices

$$\mathrm{A} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

$$\mathrm{B} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

verify that: $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, $(\mathbf{A}^T)^T = \mathbf{A}$, and $(\mathbf{B}^T)^T = \mathbf{B}$.

Problem 8

Let us define the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

Please find a nonzero matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{0}$. Does $\mathbf{BA} = \mathbf{0}$?

Problem 9

Let us define the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Please find a nonzero matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{0}$. Does $\mathbf{BA} = \mathbf{0}$?

Python Exercise 3

Let us explore some basic properties of matrices using create a matrix in NumPy:

import numpy as np

A = np.array([[2,4],[6,8]])

K= np.array([[1,0],[0,1]])

C = A + K #matrix addition

print(C)

D = A - K# matrix subtraction

print(D)

E = C@D #matrix multiplication

print(E)

A= np.array([[1,2,3], [4,5,6], [7,8,9]]) #this creates the 3 x 3 matrix called A

print(A)

print(A.shape)#prints the number of rows and columns of the matrix A

print(A.T) #prints the transpose of the matrix A

np.trace(A)# prints the trace of the matrix A

E = np.eye(2)# creates a 2 x 2 identity matrix

print(E)

F = np.diag(A)#extracts the diagonal matrix elements from the matrix A

print(F)

Now let us apply these NumPy commands in some useful exercises.

1. A n x n matrix has $\boldsymbol{n^2}$ matrix elements. For a symmetric matrix, however, not all elements are unique. Create a 2 x 2 symmetric matrix and a 3 x 3 symmetric matrix and count the total number of matrix elements and the number of possible unique matrix elements. Now work out a formula for the number of possible unique matrix elements in such a matrix.

2. Create the following arbitrary matrices: **U(4 x 1)**, **V(5 x 1)**, **W(5 x 1)**, and **A(4 x 5)**. For these matrices solve the given arithmetic problem or demonstrate why it is not solvable: $\mathbf{WU^T} + \mathbf{A^T}$.

3. Create the following arbitrary matrices: **U(5 x 1)**, **V(6 x 1)**, **W(6x 1)**, and **A(5 x 6)**. For these matrices solve the given arithmetic problem or demonstrate why it is not solvable: $\mathbf{UV^T} - \mathbf{A}$.

4. Create the following arbitrary matrices: **U(6 x 1)**, **V(7 x 1)**, **W(7 x 1)**, and **A(6 x 7)**. For these matrices solve the given arithmetic problem or demonstrate why it is not solvable: $\mathbf{VW^T} - \mathbf{A}$.

5. Create the following arbitrary matrices: **U(7 x 1)**, **V(8 x 1)**, **W(8 x 1)**, and **A(7 x 8)**. For these matrices solve the given arithmetic problem or demonstrate why it is not solvable: $\mathbf{VW^T} + \mathbf{A^T}$.

6. Create the following arbitrary matrices: $\mathbf{U}(10 \times 1)$, $\mathbf{V}(11 \times 1)$, $\mathbf{W}(11 \times 1)$, and $\mathbf{A}(10 \times 11)$. For these matrices solve the given arithmetic problem or demonstrate why it is not solvable: $\mathbf{WU^T} + \mathbf{A}$.

7. Create a matrix $\mathbf{A}$ $(\mathbf{n} \times \mathbf{n})$ and a matrix $\mathbf{B}$ $(\mathbf{n} \times \mathbf{n})$. Is the sum of their traces equal to the trace of their sums? Note the trace of a matrix only exists if that matrix is square.

8. Given the matrices of sizes $\mathbf{A}$ $(\mathbf{2} \times \mathbf{3})$, $\mathbf{B}$ $(\mathbf{3} \times \mathbf{3})$, and $\mathbf{C}$ $(\mathbf{3} \times \mathbf{4})$, determine whether each of the following operations is valid, and, if so, the size of the resulting matrix:

$$(a)\ \ CB$$

$$(b)\ \ C^TB$$

$$(c)\ \ (CB)^T$$

$$(d)\ \ C^TBC$$

$$(e)\ \ ABCB$$

$$(f)\ \ ABC$$

$$(g)\ \ C^TBA^TAC$$

$$(h)\ \ B^TBCC^TA$$

$$(i)\ \ AA^T$$

$$(j)\ \ A^TA$$

$$(k)\ \ BBA^TABBCC$$

$$(l)\ \ (CBB^TCC^T)^T$$

$$(m)\ \ (A + ACC^T B)^T A$$

$$(n)\ \ C + CA^T ABC$$

$$(o)\ \ C + BA^T ABC$$

$$(p)\ \ B + 3B + A^T A - CC^T$$

9. The nth-order **Fibonacci matrix** [named for the Italian mathematician (*circa* 1170-1250) is the n x n matrix F_n that has 1's on the main diagonal, 1's along the diagonal immediately above the main diagonal, -1's along the diagonal immediately below the main diagonal, and zeros everywhere else. Construct the sequence

$$det(F_1),\ det(F_2),\ det(F_3), ...,\ det(F_7)$$

Make a conjecture about the relationship between a term in the sequence and its two immediate predecessors, and then use your conjecture to make a guess at

$$det(F_8)$$

Check your guess by calculating this number.

10. Symmetric matrices are very useful for advanced applications in linear algebra but not all matrices are symmetric. The good news is that there is a way to convert a non-symmetric matrix into a symmetric one. Invent a matrix **A (3 x 3)** and show that you can find a new matrix **C** which is symmetric if you use the following recipie

$$C = \frac{1}{2}(A + A^T) = \frac{1}{2}(A^T + A)$$

Try it for a smaller **A (2 x 2)** matrix! The question now is this just a property of matrices of order 2 or 3, or is it true for matrices of any order? You can prove this for any matrix of order n by starting with any version of the above equation, taking the transpose of both sides of the equation, and going forth from there.

It is always useful to visualize results in science and engineering and we will certainly discuss how to plot vectors, lines, and graphs using Python later in this course. I do, however, wish to show you now how to visualize matrices in Python, which I think is very neat! You will need the Python library **matplotlib.pyplot**. You will also need the functions **plt.imshow** and **plt.show** which you should look up (i.e. google) to get a better idea of what they do. Play with your results as they are fascinating. Vary the factor **N**! You should discover something strange about your plot in terms of its matrix elements and how Python describes the positions of the matrix elements. What is it? We will return to this point in Problem Set IV.

import numpy as np

import matplotlib.pyplot as plt

N=10

B = np.random.randint(-2000,2000, size =(N,N)) #return a set of random integers between -2000 and 2000, where size is the output shape

B_symm = (B + B.T)/2

plt.imshow(B_symm, cmap ="jet") #color scheme for cmap either "gray" or "jet"

plt.show()

Problem 1

$$x_1 - 2x_2 = 3$$

$$2x_1 - 4x_2 = 6$$

$$-3x_1 + 6x_2 - 9$$

$$\underset{\sim}{A} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{pmatrix} \qquad \underset{\sim}{B} = \begin{pmatrix} 3 \\ 6 \\ -9 \end{pmatrix}$$

$$\underset{\sim}{A} | \underset{\sim}{B} = \left(\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -3 & 6 & -9 \end{array}\right) \xrightarrow[-2r_1 + r_2]{} \left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 0 \\ -3 & 6 & -9 \end{array}\right)$$

$$\xrightarrow[r_2 \text{ and } r_3]{\text{interchange}} \left(\begin{array}{cc|c} 1 & -2 & 3 \\ -3 & 6 & -9 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{3r_1 + r_2} \left(\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Corresponding set of algebraic equations

$$x_1 - 2x_2 = 3$$
$$0 \cdot x_2 = 0$$

$\Rightarrow$ x_2 can be any thing

$$x_1 = 3 + 2x_2$$

$$\boxed{\text{There are an infinite number of solutions}}$$

Problem 2

$$x_1 + x_2 + 2x_3 + x_4 = 5$$

$$2x_1 + 3x_2 - x_3 - 2x_4 = 2$$

$$4x_1 + 5x_2 + 3x_3 = 7$$

$\underset{\sim}{A} | \underset{\sim}{B}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{array}\right) \xrightarrow{-2r_1 + r_2}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & 8 \\ 4 & 5 & 3 & 0 & 7 \end{array}\right) \xrightarrow{-4r_1 + r_3} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & 8 \\ 0 & 1 & -5 & -4 & -13 \end{array}\right)$$

$$\xrightarrow{-r_2 + r_3} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & 8 \\ 0 & 0 & 0 & 0 & -21 \end{array}\right) \xrightarrow{-r_3/21}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-8r_3 + r_2} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-5r_3+r_1} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & -5 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-r_2+r_1}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 7 & 5 & 0 \\ 0 & 1 & -5 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

reduced row echelon form

Corresponding set of algebraic equations read from bottom up

$$0 \cdot x_4 = 1$$

$$x_2 - 5x_3 - 4x_4 = 0$$

$$x_1 \quad + 7x_3 + 5x_4 = 0$$

This tells us that no value for x_4 exists hence there are no solutions!!

Problem 3

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} \qquad \underset{\sim}{B} = \begin{pmatrix} -1 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$2\underset{\sim}{A} = \begin{pmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \end{pmatrix} \qquad 6\underset{\sim}{B} = \begin{pmatrix} -6 & 6 & 0 \\ 18 & 0 & 12 \\ 6 & 6 & 6 \end{pmatrix}$$

$$3\underset{\sim}{B} = \begin{pmatrix} -3 & 3 & 0 \\ 9 & 0 & 6 \\ 3 & 3 & 3 \end{pmatrix} \qquad 2\underset{\sim}{B} = \begin{pmatrix} -2 & 2 & 0 \\ 6 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix}$$ [actually not needed]

$$\underset{\sim}{C} = 2\underset{\sim}{A} - 3\underset{\sim}{B}$$

$$\underset{\sim}{C} = \begin{pmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \end{pmatrix} - \begin{pmatrix} -3 & 3 & 0 \\ 9 & 0 & 6 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -3 & -2 \\ -7 & 0 & -8 \\ -1 & -3 & -5 \end{pmatrix}$$

$$\underset{\sim}{D} = 6\underset{\sim}{B} - \underset{\sim}{A} = \begin{pmatrix} -6 & 6 & 0 \\ 18 & 0 & 12 \\ 6 & 6 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -7 & 6 & +1 \\ 17 & 0 & 13 \\ 5 & 6 & 7 \end{pmatrix}$$

Problem 4

(i) $$\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} \; ; \quad \underset{\sim}{A}^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

$$(\underset{\sim}{A}^T)^T = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \underset{\sim}{A}$$

(ii) $$\underset{\sim}{A} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} \qquad \underset{\sim}{B} = \begin{pmatrix} -1 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{B} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} -2 & 0 & -1 \\ -2 & 0 & -1 \\ -2 & 0 & -1 \end{pmatrix}$$

$$(\underset{\sim}{A}\,\underset{\sim}{B})^T = \begin{pmatrix} -2 & -2 & -2 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

$$\underset{\sim}{B}^T = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \quad ; \quad \underset{\sim}{A}^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

$$\underset{\sim}{B}^T \underset{\sim}{A}^T = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

$$\underset{\sim}{B}^T \underset{\sim}{A}^T = \begin{pmatrix} -2 & -2 & -2 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} = (\underset{\sim}{A} \underset{\sim}{B})^T$$

Problem 5

$$\underset{\sim}{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underset{\sim}{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\underset{\sim}{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{B} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} i & 0 & -i \\ 0 & -i+i & 0 \\ i & +0 & -i \end{pmatrix}$$

$$\underset{\sim}{B}\,\underset{\sim}{A} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix}$$

$$\underset{\sim}{A}\underset{\sim}{B} - \underset{\sim}{B}\underset{\sim}{A} = \frac{1}{2}\left[\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} = i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= i\underset{\sim}{C}$$

—o—

$$\underset{\sim}{B}\underset{\sim}{C} - \underset{\sim}{C}\underset{\sim}{B} = ?$$

$$\underset{\sim}{B}\underset{\sim}{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{C}\underset{\sim}{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} =$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}$$

$$\underset{\sim}{B}\,\underset{\sim}{C} - \underset{\sim}{C}\,\underset{\sim}{B} =$$

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= i\,\underset{\sim}{A}$$

$$\underset{\sim}{C}\,\underset{\sim}{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{C}\,\underset{\sim}{A} - \underset{\sim}{A}\,\underset{\sim}{C} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0&1&0\\ 0&0&0\\ 0&-1&0\end{pmatrix} - \begin{pmatrix} 0&0&0\\ 1&0&-1\\ 0&0&0\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0&1&0\\ -1&0&1\\ 0&-1&0\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(i)(-i)\begin{pmatrix} 0&1&0\\ -1&0&1\\ 0&-1&0\end{pmatrix}$$

$$= \frac{i}{\sqrt{2}}\begin{pmatrix} 0&-i&0\\ i&0&-i\\ 0&i&0\end{pmatrix} = i\,\underset{\sim}{B}$$

—∘—

$$\underset{\sim}{A}\,\underset{\sim}{A} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 0&1&0\\ 1&0&1\\ 0&1&0\end{pmatrix}\begin{pmatrix} 0&1&0\\ 1&0&1\\ 0&1&0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1&0&1\\ 0&2&0\\ 1&0&1\end{pmatrix}$$

$$\underset{\sim}{B}\,\underset{\sim}{B} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 0&-i&0\\ i&0&-i\\ 0&i&0\end{pmatrix}\begin{pmatrix} 0&-i&0\\ i&0&-i\\ 0&i&0\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1&0&-1\\ 0&2&0\\ -1&0&1\end{pmatrix}$$

$$\underset{\sim}{C}\,\underset{\sim}{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{A}^2 + \underset{\sim}{B}^2 + \underset{\sim}{C}^2 = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\right]$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Bigg]$$

$$= \frac{1}{2}\left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right] + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\underset{\sim}{I}$$

Problem 6

$$\underset{\sim}{A} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \underset{\sim}{B} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\underset{\sim}{C} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{B} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\underset{\sim}{B}\,\underset{\sim}{A} = \frac{1}{2}\left(\frac{1}{2}\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\left(\underset{\sim}{A}\,\underset{\sim}{B} - \underset{\sim}{B}\,\underset{\sim}{A}\right) = \frac{1}{4}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}\right] = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$= \frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = iC$$

$$\underset{\sim}{B}\,\underset{\sim}{C} = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\underset{\sim}{C}\,\underset{\sim}{B} = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$\underset{\sim}{B}\,\underset{\sim}{C} - \underset{\sim}{C}\,\underset{\sim}{B} = \frac{1}{4}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right]$$

$$= \frac{1}{4} \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= i\,\underset{\sim}{A}$$

$$\underset{\sim}{C}\,\underset{\sim}{A} = \frac{1}{2}\left(\frac{1}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{C} = \left(\frac{1}{2}\right)\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\underset{\sim}{C}\,\underset{\sim}{A} - \underset{\sim}{A}\,\underset{\sim}{C} = \frac{1}{4}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}(i)(-i)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= i\,\underset{\sim}{B}$$

$$\underset{\sim}{A}\,\underset{\sim}{A} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \quad \frac{1}{4} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{B}\,\underset{\sim}{B} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \quad \frac{1}{4} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{C}\,\underset{\sim}{C} = \frac{1}{2}\left(\frac{1}{2}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \quad \frac{1}{4} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{A}^2 + \underset{\sim}{B}^2 + \underset{\sim}{C}^2 = 3/4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 3/4 \; \underset{\sim}{I}$$

Problem 7

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \qquad \underset{\sim}{B} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$

$\boxed{\text{Part I}}$

$$\underset{\sim}{A}\,\underset{\sim}{B} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 2 & -4 \end{pmatrix}$$

$$(\underset{\sim}{A}\,\underset{\sim}{B})^T = \begin{pmatrix} 1 & 2 \\ 5 & -4 \end{pmatrix}$$

$$\underset{\sim}{B}^T = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \quad ; \quad \underset{\sim}{A}^T = \begin{pmatrix} 1 & 0 \\ 2 & -2 \end{pmatrix}$$

$$\underset{\sim}{B}^T \underset{\sim}{A}^T = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 5 & -4 \end{pmatrix}$$

$$= (\underset{\sim}{A}\,\underset{\sim}{B})^T$$

$\boxed{\text{Part II}}$

$$(\underset{\sim}{A}^T)^T = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \underset{\sim}{A}$$

$\boxed{\text{Part III}}$

$$(\underset{\sim}{B}^T)^T = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \underset{\sim}{B}$$

Problem 8

Given $\underset{\sim}{A} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

find $\underset{\sim}{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\underset{\sim}{A}\underset{\sim}{B} = \underset{\sim}{0}$

$$\underset{\sim}{A}\,\underset{\sim}{B} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 3a+6c & 3b+6d \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \underset{\sim}{0}$$

$a + 2c = 0$ ①

$3a + 6c = 0$ ②

$b + 2d = 0$ ③

$3b + 6d = 0$ ④

Equations ① & ② say the same thing so

$a = -2$
$c = 1$ and multiplies of them

Equations ③ & ④ say the same thing so

$b = 2$
$d = -1$ and multiplies of them

$$\underset{\sim}{B} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{B} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \underset{\sim}{0}$$

Does $\underset{\sim}{B}\,\underset{\sim}{A} = \underset{\sim}{0}$?

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ -2 & -4 \end{pmatrix} \neq \underset{\sim}{0}$$

No !!

Problem 9

$$\underset{\sim}{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Find $\underset{\sim}{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\underset{\sim}{A}\,\underset{\sim}{B} = \underset{\sim}{0}$$

$$\underset{\sim}{A}\,\underset{\sim}{B} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underset{\sim}{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+c & b+d \\ a+2c & b+2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{A}\,\underset{\sim}{B} = \underset{\sim}{0} \Rightarrow \begin{aligned} a+c &= 0 \quad (1) \\ b+d &= 0 \quad (2) \\ a+2c &= 0 \quad (3) \\ b+2d &= 0 \quad (4) \end{aligned}$$

Solving (2) + (4)

$$\Rightarrow -d = 0$$

$$b = 0$$

Solving (1) + (3)

$$\Rightarrow a+c = 0 = a + -\frac{a}{2} = \frac{a}{2} = 0$$

$$c = 0$$

All of these are trivial solutions

For non-trivial solutions

$① \rightarrow c = -a$

but ③ says

$$a + 2c = a + 2(-a) = -a = 0$$

so there is no non-zero value for a!

The same argument goes for c

$② \rightarrow b = -d$

but ④ says

$$b + 2d = -d + 2d = d = 0$$

There is no non-zero value for d and the same argument goes for b!

Alternatively

$a = -c$ only works for $a = c = 0$

$b = -d$ only works for $b = d = 0$

No non-zero matrix $\underset{\sim}{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ exists

such that

$$\underset{\sim}{A}\,\underset{\sim}{B} = \underset{\sim}{0}$$

where $\underset{\sim}{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

What about

$$\underset{\sim}{B}\,\underset{\sim}{A} \; ?$$

$$\underset{\sim}{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \underset{\sim}{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\underset{\sim}{B}\,\underset{\sim}{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} a+b & a+b \\ c+d & c+2d \end{pmatrix}$$

$$\overset{?}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$a+b=0$ ①

$a+b=0$ ②

$c+d=0$ ③

$c+2d=0$ ④

Equations ① + ② imply

$a=-b$

[The only way that this is true, if $a=0$ is as $2 \neq -2$, $10^4 \neq -10^4$ etc...

Equations ③ + ④ imply

$c=0$ and $d=0$

There is no non-zero matrix $\underset{\sim}{B}$ such that

$\underset{\sim}{B}\,\underset{\sim}{A} = \underset{\sim}{0}$ where $\underset{\sim}{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$